King Fahd University of Petroleum & Minerals
(KFUPM)

Department of Mathematics

Course Title: AS289

Midterm Exam – Term 252

Professor: Kroumi Dhaker

Question 1.

Eric deposits 100 into a savings account at time 0, which pays interest at a nominal rate of i, compounded semiannually.

Mike deposits 200 into a different savings account at time 0, which pays simple interest at an annual rate of i.

Eric and Mike earn the same amount of interest during the last 6 months of the 8th year. Calculate i.

(A) 9.06%

(B) 9.26%

(C) 9.46%

(D) 9.66%

(E) 9.86%

Solution.

Let the nominal rate convertible semiannually be i, so the half-year rate is $i/2$. Eric's balance at $t = 7.5$ (i.e., after 15 half-years) is $100(1+i/2)^{15}$, so the interest earned in the last half-year is

$$100(1 + i/2)^{15} \cdot \frac{i}{2}.$$

Mike earns simple interest $200i$ per year, hence in the last half-year he earns $200i \cdot \frac{1}{2} = 100i$. Equating the two interest amounts and canceling $100i$ gives

$$(1 + i/2)^{15} = 2 \quad \Rightarrow \quad i = 2\left(2^{1/15} - 1\right) \approx 0.0946 = 9.46\%.$$

Question 2.

Ernie makes deposits of 100 at time 0, and X at time 3. The fund grows at a force of interest

$$\delta_t = \frac{t^2}{100}, \quad t > 0.$$

The amount of interest earned from time 3 to time 6 is also X. Calculate X.

(A) 385

(B) 485

(C) 585

(D) 685

(E) 785

Solution.

With force of interest $\delta_t = t^2/100$, the accumulation factor is

$$A(s,t) = \exp\Big(\int_s^t \delta_u \, du\Big) = \exp\left(\int_s^t \frac{u^2}{100} \, du\right) = \exp\left(\frac{t^3 - s^3}{300}\right).$$

So $A(0,3) = e^{27/300} = e^{9/100}$ and $A(3,6) = e^{(216-27)/300} = e^{63/100}$. Balance at time 3 is $100e^{9/100} + X$. Interest earned on $[3,6]$ equals

$$(100e^{9/100} + X)\Big(A(3,6) - 1\Big) = (100e^{9/100} + X)\left(e^{63/100} - 1\right).$$

Set this equal to X and solve:

$$(100e^{9/100} + X)\left(e^{63/100} - 1\right) = X \Rightarrow X = \frac{100e^{9/100}\left(e^{63/100} - 1\right)}{2 - e^{63/100}} \approx 784.6 \approx 785.$$

Question 3.

At a nominal interest rate of i convertible semi-annually, an investment of 1000 immediately and 1500 at the end of the first year will accumulate to 2600 at the end of the second year. Calculate i.

(A) 2.75%

(B) 2.77%

(C) 2.79%

(D) 2.81%

(E) 2.83%

Solution.

With nominal rate i convertible semiannually, the semiannual accumulation factor is $1 + \frac{i}{2}$. Accumulating to time 2 (four half-years) gives

$$1000\left(1 + \frac{i}{2}\right)^4 + 1500\left(1 + \frac{i}{2}\right)^2 = 2600.$$

Let $t = \left(1 + \frac{i}{2}\right)^2$. Then

$$1000t^2 + 1500t - 2600 = 0 \quad \Rightarrow \quad t = \frac{-1500 + \sqrt{1500^2 + 4(1000)(2600)}}{2000} \approx 1.0283.$$

Hence $1 + \frac{i}{2} = \sqrt{t}$ and

$$i = 2(\sqrt{t} - 1) \approx 0.0281 = 2.81\%.$$

Question 4.

A store is running a promotion during which customers have two options for payment.

- Option one is to pay 90% of the purchase price two months after the date of sale.
- Option two is to deduct $X\%$ off the purchase price and pay cash on the date of sale.

A customer wishes to determine X such that he is indifferent between the two options when valuing them using an effective annual interest rate of 8%.

Which of the following equations of value would the customer need to solve?

(A) $\left(\frac{X}{100}\right)\left(1+\frac{0.08}{6}\right) = 0.90$

(B) $\left(1-\frac{X}{100}\right)\left(1+\frac{0.08}{6}\right) = 0.90$

(C) $\left(\frac{X}{100}\right)(1.08)^{1/6} = 0.90$

(D) $\left(\frac{X}{100}\right)\left(\frac{1.08}{1.06}\right) = 0.90$

(E) $\left(1-\frac{X}{100}\right)(1.08)^{1/6} = 0.90$

Solution.

Let the purchase price be P. Paying cash now costs $(1-X/100)P$. The deferred option pays $0.90P$ at $t=\frac{2}{12}=\frac{1}{6}$ years, whose present value is $0.90P(1.08)^{-1/6}$. Indifference requires

$$(1-X/100)P = 0.90P(1.08)^{-1/6} \quad \Longleftrightarrow \quad (1-X/100)(1.08)^{1/6} = 0.90.$$

Question 5.

A bank offers the following choices for certificates of deposit:

Term (in years)	Nominal annual interest rate convertible quarterly
1	4.00%
3	5.00%
5	5.65%

The certificates mature at the end of the term. The bank does NOT permit early withdrawals. During the next 6 years the bank will continue to offer certificates of deposit with the same terms and interest rates.

An investor initially deposits 10,000 in the bank and withdraws both principal and interest at the end of 6 years. Calculate the maximum annual effective rate of interest the investor can earn over the 6-year period.

(A) 5.09%

(B) 5.22%

(C) 5.35%

(D) 5.48%

(E) 5.61%

Solution.

If a certificate has term T years and nominal rate j convertible quarterly, its T-year accumulation factor is

$$(1 + j/4)^{4T}.$$

Over 6 years, the best strategies are:
(i) 1-year then 5-year:

$$A_{1+5} = \left(1 + \frac{0.04}{4}\right)^4 \left(1 + \frac{0.0565}{4}\right)^{20} \approx 1.37757507.$$

(ii) 3-year then 3-year:

$$A_{3+3} = \left(1 + \frac{0.05}{4}\right)^{12} \left(1 + \frac{0.05}{4}\right)^{12} = \left(1 + \frac{0.05}{4}\right)^{24} \approx 1.34735.$$

Since $A_{1+5} > A_{3+3}$, the maximum 6-year growth factor is A_{1+5}, so the maximum annual effective rate is

$$i = \sqrt[6]{A_{1+5}} - 1 \approx \sqrt[6]{1.37757507} - 1 \approx 0.05484 = 5.48\%.$$

Question 6.

The parents of three children, ages 1, 3, and 6, wish to set up a trust fund that will pay X to each child upon attainment of age 18, and Y to each child upon attainment of age 21.

They will establish the trust fund with a single investment of Z.

Which of the following is the correct equation of value for Z?

(A) $\dfrac{X}{v^{17}+v^{15}+v^{12}}+\dfrac{Y}{v^{20}+v^{18}+v^{15}}$

(B) $3\left[Xv^{18}+Yv^{21}\right]$

(C) $3Xv^{3}+Y\left[v^{20}+v^{18}+v^{15}\right]$

(D) $(X+Y)\dfrac{v^{20}+v^{18}+v^{15}}{v^{3}}$

(E) $X\left[v^{17}+v^{15}+v^{12}\right]+Y\left[v^{20}+v^{18}+v^{15}\right]$

Solution.

From ages 1, 3, 6 to age 18 the times are 17, 15, 12 years, so the present value of the X payments is

$$X(v^{17}+v^{15}+v^{12}).$$

To age 21 the times are 20, 18, 15 years, so the present value of the Y payments is

$$Y(v^{20}+v^{18}+v^{15}).$$

Thus

$$Z=X(v^{17}+v^{15}+v^{12})+Y(v^{20}+v^{18}+v^{15}).$$

Question 7.

A perpetuity costs 77.1 and makes annual payments at the end of the year.

The perpetuity pays 1 at the end of year 2, 2 at the end of year 3, $\ldots$, n at the end of year $(n+1)$. After year $(n+1)$, the payments remain constant at n.

The annual effective interest rate is 10.5%. Calculate n.

(A) 17

(B) 18

(C) 19

(D) 20

(E) 21

Solution.

Let $i = 0.105$ and $v = (1+i)^{-1}$.
Consider the *increasing perpetuity* that pays 1 at time 2, 2 at time 3, 3 at time 4, and so on forever. Its present value is

$$PV_1 = \sum_{k=1}^{\infty} k\, v^{k+1} = v\left(\frac{1}{i} + \frac{1}{i^2}\right) = \frac{1}{i^2}.$$

Our contract matches this increasing perpetuity up to payment n at time $n+1$, and then *levels off* at n. The difference between the increasing perpetuity and our contract is another increasing perpetuity shifted n years, so its present value is

$$PV_2 = PV_1\, v^n.$$

Therefore

$$77.1 = PV_1 - PV_2 = PV_1(1 - v^n) = \frac{1}{i^2}(1 - v^n).$$

Solve for v^n:

$$v^n = 1 - 77.1\, i^2 = 1 - 77.1(0.105)^2 \approx 0.1499725.$$

Hence

$$n = \frac{\ln(v^n)}{\ln(v)} \approx \frac{\ln(0.1499725)}{\ln\left(\frac{1}{1.105}\right)} \approx 19.00.$$

Question 8.

A perpetuity-immediate pays 100 per year. Immediately after the fifth payment, the perpetuity is exchanged for a 25-year annuity-immediate that will pay X at the end of the first year. Each subsequent annual payment will be 8% greater than the preceding payment. The annual effective rate of interest is 8%.

Calculate X.

(A) 54

(B) 64

(C) 74

(D) 84

(E) 94

Solution.

Immediately after the 5th payment (time 5), the value of the remaining perpetuity is

$$\frac{100}{0.08} = 1250.$$

The 25-year annuity has payments X, $1.08X$, $1.08^2X, \ldots,$ and since the growth rate equals the interest rate,

$$PV_5 = \sum_{k=1}^{25} X(1.08)^{k-1}v^k = \sum_{k=1}^{25} X\frac{1}{1.08} = \frac{25X}{1.08}.$$

Set $\dfrac{25X}{1.08} = 1250$ to get $X = 54$.

Question 9.

To accumulate 8000 at the end of $3n$ years, deposits of 98 are made at the end of each of the first n years and 196 at the end of each of the next $2n$ years.

The annual effective rate of interest is i. You are given $(1+i)^n = 2$.

Determine i.

(A) 11.25%

(B) 11.75%

(C) 12.25%

(D) 12.75%

(E) 13.25%

Solution.

Note that the future value is given by

$$\begin{aligned}
8000 &= 98 s_{\overline{n}|\, i}(1+i)^{2n} + 196 s_{\overline{2n}|\, i} \\
&= 98\frac{(1+i)^n - 1}{i}(1+i)^{2n} + 196\frac{(1+i)^{2n}-1}{i} \\
&= 98\frac{2-1}{i}2^2 + 196\frac{2^2-1}{i} \\
&= \frac{980}{i},
\end{aligned}$$

so

$$i = \frac{980}{8000} = 0.1225 = 12.25\%.$$

Question 10.

An insurance company has an obligation to pay the medical costs for a claimant. Average annual claims costs today are \$5,000, and medical inflation is expected to be 7% per year. The claimant is expected to live an additional 20 years. Claim payments are made at yearly intervals, with the first claim payment to be made one year from today.

Find the present value of the obligation if the annual interest rate is 5%.

(A) 87,932

(B) 102,514

(C) 114,611

(D) 122,634

(E) Cannot be determined

Solution.

It is a geometric annuity immediate where $i = 5\%$ and $g = 7\%$ with the first payment is $5000 \times 1.07 = 5350$. The present value is

$$PV = 5350\frac{1-\left(\frac{1+g}{1+i}\right)^{20}}{i-g} = 5350\frac{1-(1.07/1.05)^{20}}{0.05-0.07} \approx 122{,}634.$$

Question 11.

An annuity pays 1 at the end of each year for n years. Using an annual effective interest rate of i, the accumulated value of the annuity at time $(n+1)$ is 13.776. It is also known that $(1+i)^n = 2.476$.

Calculate n.

(A) 4

(B) 5

(C) 6

(D) 7

(E) 8

Solution.

The value at time n is

$$s_{\overline{n|}\,i} = \frac{(1+i)^n - 1}{i} = \frac{(2.476 - 1}{i} = \frac{1.476}{i}.$$

Accumulating one more year gives

$$(1+i)s_{\overline{n|}\,i} = \frac{1.476(1+i)}{i} = 13.776.$$

Solving this equation leads to $i = 0.12$. As $(1+i)^n = 2.476$, we obtain

$$n = \frac{\ln 2.476}{\ln(1+i)} \approx 8.$$

Question 12.

A 20-year loan of 1000 is repaid with payments at the end of each year. Each of the first ten payments equals 150% of the amount of interest due. Each of the last ten payments is X. The lender charges interest at an annual effective rate of 10%. Calculate X.

(A) 32

(B) 57

(C) 70

(D) 97

(E) 117

Solution.

In year $t \leq 10$, payment $K_t = 1.5I_t = 1.5 \times 0.10\text{OB}_{t-1} = 0.15\text{OB}_{t-1}$. Then, the principal repaid is

$$PR_t = K_t - I_t = 0.15\text{OB}_{t-1} - 0.10\text{OB}_{t-1} = 0.05\text{OB}_{t-1},$$

so

$$\text{OB}_t = \text{OB}_{t-1} - PR_t = 0.95\text{OB}_{t-1}.$$

Thus

$$\text{OB}_{10} = 0.95^{10} \times \text{OB}_0 = 0.95^{10} \times 1000 \approx 598.737.$$

This is amortized over 10 more years with level payments X at 10%:

$$X = \frac{\text{OB}_{10}}{a_{\overline{10|}\ 0.1}} = 598.737 \times \frac{0.10}{1 - 1.10^{-10}} \approx 97.44 \approx 97.$$

Question 13.

A 10-year loan of 2000 is to be repaid with payments at the end of each year. It can be repaid under the following two options:

(i) Equal annual payments at an annual effective rate of 8.07%.

(ii) Installments of 200 each year plus interest on the unpaid balance at an annual effective rate of i.

The sum of the payments under option (i) equals the sum of the payments under option (ii).

Determine i.

(A) 8.75%

(B) 9.00%

(C) 9.25%

(D) 9.50%

(E) 9.75%

Solution.

Option (i): Each payment is

$$P = \frac{2000}{a_{\overline{10}|0.0807}} = 299$$

so that the sum of payments is $10P \approx 2990$.
Option (ii): We have $\text{OB}_t = 2000 - 200t$ so that

$$I_t = i \times \text{OB}_{t-1} = i(2000 - 200(t-1)).$$

Then, the sum of the payments is

$$\begin{aligned}\text{OB}_0 + \sum_{t=1}^{10} I_t &= 2000 + \sum_{t=1}^{10} i(2000 - 200(t-1)) = 2000 + i\frac{(2000+200)10}{2} \\ &= 2000 + 11000i.\end{aligned}$$

Set $2990 \approx 2000 + 11000i$ giving $i \approx 0.09 = 9.00\%$.

Question 14.

A loan is amortized over five years with monthly payments at a nominal interest rate of 9% compounded monthly. The first payment is 1000 and is to be paid one month from the date of the loan. Each succeeding monthly payment will be 2% lower than the prior payment.

Calculate the outstanding loan balance immediately after the 40th payment is made.

(A) 6751

(B) 6889

(C) 6941

(D) 7030

(E) 7344

Solution.

The nominal rate is 9% compounded monthly, so the monthly rate is

$$i = \frac{0.09}{12} = 0.0075, \qquad v = \frac{1}{1+j} = \frac{1}{1.0075}.$$

Immediately after the 40th payment, the outstanding balance is the present value at time 40 of the remaining 20 payments. Note that these payments form a geometric annuity immediate with ratio $n = 20$, $i = 0.0075$, $g = -0.02$, and the first payment is $1000 \times (1+g)^{40}$ (payment at time 41). Then, we have

$$\text{OB}_{40} = 1000 \times (1+g)^{40} \times \frac{1 - \left(\dfrac{1+g}{1+i}\right)^{20}}{i-g} \approx 6889$$

Question 15.

Ron is repaying a loan with payments of 1 at the end of each year for n years. The amount of interest paid in period t plus the amount of principal repaid in period $t+1$ equals X.

Calculate X.

(A) $1 + \dfrac{v^{n-1}}{i}$

(B) $1 + \dfrac{v^{n-1}}{d}$

(C) $1 + v^{n-t}i$

(D) $1 + v^{n-t}d$

(E) $1 + v^{n-t}$

Solution.

We have $I_t = 1 - v^{n-t+1}$ and $PR_{t+1} = v^{n-t}$ so that

$$X = I_t + PR_{t+1} = 1 - v^{n-t+1} + v^{n-t} = 1 + v^{n-t}(1-v) = 1 + v^{n-t}d,$$

since $d = 1 - v$.

Question 16.

A bank customer takes out a loan of 500 with a 16% nominal interest rate convertible quarterly. The customer makes payments of 20 at the end of each quarter.

Calculate the amount of principal in the fourth payment.

(A) 0.0

(B) 0.9

(C) 2.7

(D) 5.2

(E) There is not enough information to calculate the amount of principal.

Solution.

The quarterly rate is $0.16/4 = 0.04$. Interest each quarter on a 500 balance is $500(0.04) = 20$, which equals the payment. Hence each payment is entirely interest and the principal component is 0, including the 4th payment.

Question 17.

Hank purchases a \$200,000 home. Mortgage payments are to be made monthly for 30 years with the first payment to be made one month from the loan origination. The annual effective rate of interest is 5%.

Starting with the 100th payment, 400 is added to each payment in order to repay the mortgage earlier. What would be the amount of the last payment.

(A) 565

(B) 567

(C) 1020

(D) 1060

(E) 1460

Solution.

Monthly effective rate is $j = (1.05)^{1/12} - 1 = 0.004074$. Level payment for a 30-year (360-month) mortgage is

$$R = \frac{200000}{a_{\overline{360}|j}} \approx 1060.09$$

After 99 payments, the outstanding balance is

$$\text{OB}_{99} = \text{OB}_0 \times (1+j)^{99} - Rs_{\overline{99}|\ j} \approx 170162.19.$$

From payment 100 onward, the payment is $C = R + 400 = 1460.09$. Now, we will fix the number of remaining payments n. As

$$\text{OB}_{99} = Ca_{\overline{n}|j} = C\frac{1-(1+j)^{-n}}{j} \text{ so that } (1+j)^{-n} = 1 - \frac{j\text{OB}_{99}}{C}$$

Applying ln on both sides will lead to

$$n = -\frac{\ln\left(1 - \dfrac{j\text{OB}_{99}}{C}\right)}{\ln(1+j)} \approx 158.39.$$

This means that we need 158 payments of C and a last payment of amount less than C. After 158 such payments, the remaining balance is about

$$\tilde{\text{OB}}_{158} = 170162.19 \times (1+j)^{158} - 1460.09s_{\overline{158}|j} = 564.85,$$

so the last payment (one month later) is

$$\text{Last} = 564.85(1+j) \approx 567.195.$$

Question 18.

A loan is repaid with level annual payments based on an annual effective interest rate of 7%. The 8th payment consists of 789 of interest and 211 of principal.

Calculate the amount of interest paid in the 18th payment.

(A) 415

(B) 444

(C) 556

(D) 585

(E) 612

Solution.

Level payment is 789+211 = 1000. Interest in the 8th payment is $0.07\mathrm{OB}_7 = 789$, so $\mathrm{OB}_7 = 789/0.07 = 11271.4286$.
As

$$\mathrm{OB}_{17} = \mathrm{OB}_7 \times 1.07^{10} - 1000\, s_{\overline{10}|0.07} \approx 8356.16$$

Thus interest in the 18th payment is

$$I_{18} = 0.07 \times \mathrm{OB}_{17} \approx 584.93 \approx 585.$$

Question 19.

You are given the following information about a loan of L that is to be repaid with a series of 16 annual payments:

- The first payment of 2000 is due one year from now.
- The next seven payments are each 3% larger than the preceding payment.
- From the 9th to the 16th payment, each payment will be 3% less than the preceding payment.
- The loan has an annual effective interest rate of 7%.

Calculate L.

(A) 20,689

(B) 20,716

(C) 20,775

(D) 21,147

(E) 22,137

Solution.

The first eight payments form a geometric immediate annuity with $i = 0.07$, $g_1 = 0.03$ and the first payment is 2000. Its present value is

$$PV_1 = 2000 \frac{1 - \left(\frac{1+g_1}{1+i}\right)^n}{i - g_1} = 13136.41.$$

The next eight payments are the discount present value of a second geometric annuity immediate with $i = 0.7$, $g_2 = -0.03$ and the first payment is $2000 \times 1.03^7 \times 0.97 = 2386$. Then its value at time 0 is

$$PV_2 = 2386 \frac{1 - \left(\frac{1+g_2}{1+i}\right)^n}{i - g_2} \times v^{-8} = 7552$$

Therefore the loan amount is the present value

$$L = PV_1 + PV_2 \approx 20688.77.$$