King Fahd University of Petroleum & Minerals
(KFUPM)

Department of Mathematics

Course Title: AS289 Financial Mathematics

Final Exam - Multiple Choice Solutions

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Question 1.

You are given the following information with respect to a bond:

(i) par value: 1000

(ii) term to maturity: 3 years

(iii) annual coupon rate: 6% payable annually

You are also given that the one, two, and three year annual spot interest rates are 7%, 8%, and 9% respectively.

Calculate the value of the bond.

(A) 906

(B) 926

(C) 930

(D) 950

(E) 1000

Solution.

The bond pays 60 at the end of years 1 and 2, and 1060 at the end of year 3. Discounting each cash flow using the corresponding spot rate gives

$$P = \frac{60}{1.07} + \frac{60}{1.08^2} + \frac{1060}{1.09^3} = 926.03.$$

Thus the value of the bond is 926.

Question 2.

A 10,000 par value 10-year bond with 8% annual coupons is bought at a premium to yield an annual effective rate of 6%.

Calculate the interest portion of the 7th coupon.

(A) 632

(B) 642

(C) 651

(D) 660

(E) 667

Solution.

The book value at time 6 is the present value of future payments:

$$BV_6 = 10{,}000v^4 + 800a_{\overline{4}|0.06} = 7920.94 + 2772.08 = 10{,}693.$$

The interest portion of the 7th coupon is

$$10{,}693(0.06) = 641.58.$$

Thus the answer is 642.

Question 3.

You have decided to invest in Bond X, an n-year bond with semi-annual coupons and the following characteristics:

(i) Par value is 1000.

(ii) The ratio of the semi-annual coupon rate, r, to the desired semi-annual yield rate, i, is 1.03125.

(iii) The present value of the redemption value is 381.50.

Given $(1+i)^{-n} = 0.5889$, calculate the price of bond X.

(A) 1019

(B) 1029

(C) 1050

(D) 1055

(E) 1072

Solution.

Let C be the redemption value and let $v = (1+i)^{-1}$. Since the present value of the redemption value is 381.50,

$$Cv^{2n} = 381.50.$$

Also,

$$v^{2n} = (0.5889)^2.$$

The price is

$$P = 1000ra_{\overline{2n}|i} + Cv^{2n} = 1000\frac{r}{i}(1 - v^{2n}) + 381.50.$$

Using $r/i = 1.03125$ gives

$$P = 1000(1.03125)(1 - 0.5889^2) + 381.50 = 1055.11.$$

Thus the price is 1055.

Question 4.

As of 12/31/2013, an insurance company has a known obligation to pay 1,000,000 on 12/31/2017. To fund this liability, the company immediately purchases 4-year 5% annual coupon bonds totaling 822,703 of par value. The company anticipates reinvestment interest rates to remain constant at 5% through 12/31/2017. The maturity value of the bond equals the par value.

Consider two reinvestment interest rate movement scenarios effective 1/1/2014. Scenario A has interest rates drop by 0.5%. Scenario B has interest rates increase by 0.5%.

Determine which of the following best describes the insurance company's profit or loss as of 12/31/2017 after the liability is paid.

(A) Scenario A – 6,610, Scenario B – 11,150

(B) Scenario A – (14,760), Scenario B – 14,420

(C) Scenario A – (18,910), Scenario B – 19,190

(D) Scenario A – (1,310), Scenario B – 1,320

(E) Scenario A – 0, Scenario B – 0

Solution.

Under either scenario, the company will have

$$822{,}703(0.05) = 41{,}135$$

to invest at the end of each of the four years. Under Scenario A, these payments will be invested at 4.5% and accumulate to

$$41{,}135 s_{\overline{4}|0.045} = 41{,}135(4.2782) = 175{,}984.$$

Adding the maturity value produces

$$175{,}984 + 822{,}703 = 998{,}687,$$

for a loss of 1,313. Thus only answer D has this value.

The Scenario B calculation is

$$41{,}135 s_{\overline{4}|0.055} = 41{,}135(4.3423) = 178{,}621,$$

and

$$178{,}621 + 822{,}703 - 1{,}000{,}000 = 1{,}324.$$

Thus the answer is D.

Question 5.

Matt purchased a 20-year par value bond with an annual nominal coupon rate of 8% payable semiannually at a price of 1722.25. The bond can be called at par value X on any coupon date starting at the end of year 15 after the coupon is paid. The lowest yield rate that Matt can possibly receive is a nominal annual interest rate of 6% convertible semiannually.

Calculate X.

(A) 1400

(B) 1420

(C) 1440

(D) 1460

(E) 1480

Solution.

The coupon rate per half-year is 4%, and the yield rate per half-year is 3%. Since this is a premium bond, the lowest yield occurs if the bond is called at the earliest possible date, at the end of year 15, after 30 coupon periods.

The equation of value is

$$1722.25 = 0.04X a_{\overline{30}|0.03} + X v_{0.03}^{30}.$$

Thus

$$1722.25 = X\left(0.04 a_{\overline{30}|0.03} + v_{0.03}^{30}\right) = 1.196X.$$

Therefore

$$X = 1440.$$

Question 6.

A company is required to pay 190,000 in 20.5 years. The company creates an investment portfolio using three bonds with annual coupons, so that its position is Redington immunized based on an annual effective interest rate of 7%. The table below shows the Macaulay duration for each of the bonds.

	Macaulay Duration
Bond A	10 years
Bond B	15 years
Bond C	30 years

The company invests twice as much money in Bond C as in Bond B.

Calculate the amount the company invests in Bond A.

(A) 6,640

(B) 8,630

(C) 11,075

(D) 13,308

(E) 14,240

Solution.

Let x be the amount invested in Bond A and y the amount invested in Bond B. Then $2y$ is invested in Bond C. To match the present value of the assets and liabilities,

$$x + y + 2y = 190{,}000(1.07)^{-20.5},$$

so

$$x + 3y = 47{,}466.39.$$

To match the Macaulay durations,

$$20.5 = \frac{10x + 15y + 30(2y)}{47{,}466.39}.$$

Then

$$20.5(47{,}466.39) = 10(47{,}466.39 - 3y) + 75y,$$

so

$$y = \frac{20.5(47{,}466.39) - 10(47{,}466.39)}{45} = 11{,}075.49.$$

Therefore

$$x = 47{,}466.39 - 3(11{,}075.49) = 14{,}239.92.$$

The amount invested in Bond A is 14,240.

Question 7.

A company is considering investing in a particular project. The project requires an investment of X today. Additional investments are required at the beginning of each of the next five years, with each year's investment 5% greater than the previous year's investment.

The investment is expected to produce an income of 100 per year at the end of each year forever, with the first payment expected at the end of the first year.

At an annual effective interest rate of 10.25%, the project has a net present value of zero. Calculate X.

(A) 183

(B) 192

(C) 205

(D) 215

(E) 225

Solution.

The present value of the perpetual income is

$$\frac{100}{0.1025} = 975.61.$$

The investment payments occur at times $0, 1, 2, 3, 4, 5$ and have present value

$$X\left[1 + \frac{1.05}{1.1025} + \left(\frac{1.05}{1.1025}\right)^2 + cdots + \left(\frac{1.05}{1.1025}\right)^5\right].$$

Since $1.1025 = (1.05)^2$, this becomes

$$X(1 + 1.05^{-1} + 1.05^{-2} + \cdots + 1.05^{-5}) = 5.3295X.$$

Thus

$$5.3295X = 975.61,$$

so

$$X = 183.06.$$

The answer is 183.

Question 8.

A common stock pays dividends at the end of each year into perpetuity. Assume that the dividend increases by 2% each year.

Using an annual effective interest rate of 5%, calculate the Macaulay duration of the stock in years.

(A) 27

(B) 35

(C) 44

(D) 52

(E) 58

Solution.

The duration is

$$\frac{\sum_{t=1}^{\infty} tv^t R_t}{\sum_{t=1}^{\infty} v^t R_t}.$$

Since the dividend increases by 2% each year, this ratio is equivalent to the duration of a level perpetuity at rate j such that

$$(1+j)^{-1} = \frac{1.02}{1.05}.$$

Thus

$$j = \frac{1.05}{1.02} - 1 = \frac{0.03}{1.02}.$$

For a perpetuity-immediate, the Macaulay duration is

$$\frac{1}{d_j} = \frac{1+j}{j}.$$

Therefore

$$D = \frac{1+j}{j} = \frac{1.05/1.02}{0.03/1.02} = \frac{1.05}{0.03} = 35.$$

Question 9.

An investor pays 4000 today for a three-year investment that returns cash flows of 1400 at the end of each year. The cash flows can be reinvested at the positive annual effective interest rate of i. Using an annual effective rate of interest of 4%, the net present value of this investment is 0.

Calculate i.

(A) 2.5%

(B) 3.0%

(C) 3.5%

(D) 4.0%

(E) 7.0%

Solution.

At time 3, the accumulated value of the initial investment at 4% is

$$4000(1.04)^3.$$

The accumulated value of the three cash flows at reinvestment rate i is

$$1400(1+i)^2 + 1400(1+i) + 1400.$$

Thus

$$4000(1.04)^3 = 1400\left[(1+i)^2 + (1+i) + 1\right].$$

Solving gives the positive root

$$i = 0.0697.$$

Thus the answer is 7.0%.

Question 10.

Consider an annuity-immediate with 30 annual payments of 250. If the annual effective interest rate is 11%, then

$$P(0.11) = 250a_{\overline{30}|0.11} = 2173.4481,$$

and

$$D_{\text{mac}}(0.11) = \frac{(Ia)_{\overline{30}|0.11}}{a_{\overline{30}|0.11}} = 8.7206.$$

Use the first-order modified approximation to estimate the price of the annuity if $i = 0.10$.

(A) 2173.45

(B) 2250.60

(C) 2344.20

(D) 2355.98

(E) 2356.73

Solution.

From the given Macaulay duration,

$$D_{\text{mod}}(0.11) = \frac{D_{\text{mac}}(0.11)}{1 + 0.11} = \frac{8.7206}{1.11} = 7.8564.$$

Using the first-order modified approximation,

$$P(0.10) \approx P(0.11) - D_{\text{mod}}(0.11)P(0.11)(0.10 - 0.11).$$

Therefore

$$P(0.10) \approx 2173.4481 - 7.8564(2173.4481)(-0.01) = 2344.2021.$$

Thus the estimated price is 2344.20.

Question 11.

You are given the following term structure of interest rates:

Length of investment in years	Spot rate
1	7.50%
2	8.00%
3	8.50%
4	9.00%
5	9.50%

Calculate the one-year annual effective rate for the fifth year implied by this term structure.

(A) 9.0%

(B) 9.5%

(C) 10.0%

(D) 10.5%

(E) 11.5%

Solution.

The one-year rate for the fifth year is the one-year forward rate from time 4 to time 5. It satisfies

$$(1.09)^4(1+f) = (1.095)^5.$$

Thus

$$f = \frac{1.095^5}{1.09^4} - 1 = 0.1152.$$

The answer is 11.5%.

Question 12.

You are given the following term structure of interest rates:

Length of investment in years	Spot rate
1	7.50%
2	8.00%
3	8.50%
4	9.00%
5	9.50%
6	10.00%

Calculate the one-year forward rate, deferred four years, implied by this term structure.

(A) 9.5%

(B) 10.0%

(C) 11.5%

(D) 12.0%

(E) 12.5%

Solution.

A one-year forward rate deferred four years is the one-year rate from time 4 to time 5. Thus

$$(1.09)^4(1+f) = (1.095)^5.$$

Solving,

$$f = \frac{1.095^5}{1.09^4} - 1 = 0.1152.$$

Thus the answer is 11.5%.

Question 13.

A project requires an initial capital outlay of 30,000 and will return the following amounts at the ends of the next 5 years:

$$14{,}000, \quad 12{,}000, \quad 6{,}000, \quad 4{,}000, \quad 2{,}000.$$

Let j be the internal rate of return and X be the net present value based on a cost of capital of 10% per year. Then $(j, X) =$

(A) 11.47% and 1016.41

(B) 11.47% and 1126.41

(C) 12.03% and 1346.41

(D) 12.03% and 1126.41

(E) 13.41% and 1016.41

Solution.

The net present value at a cost of capital of 10% is

$$X = -30000 + \frac{14000}{1.10} + \frac{12000}{(1.10)^2} + \frac{6000}{(1.10)^3} + \frac{4000}{(1.10)^4} + \frac{2000}{(1.10)^5} \approx 1126.41.$$

The internal rate of return j satisfies

$$-30000 + \frac{14000}{1+j} + \frac{12000}{(1+j)^2} + \frac{6000}{(1+j)^3} + \frac{4000}{(1+j)^4} + \frac{2000}{(1+j)^5} = 0.$$

Solving this equation numerically gives

$$j \approx 0.120259 = 12.03\%.$$

Thus

$$(j, X) = (12.03\%, 1126.41).$$

Question 14.

An investor purchases a 1200 face amount zero-coupon bond for a price of 1000. With respect to the bond's annual effective yield rate, the Macaulay duration is four years and the modified duration is d years.

Calculate d.

(A) 3.33

(B) 3.82

(C) 3.86

(D) 4.00

(E) 4.19

Solution.

For a zero-coupon bond, the Macaulay duration is the time to maturity, so the maturity is 4 years. The yield rate satisfies

$$1000 = 1200(1+i)^{-4}.$$

Thus

$$1+i = \left(\frac{1200}{1000}\right)^{1/4} = 1.046635.$$

The modified duration is

$$d = \frac{D_{\text{mac}}}{1+i} = \frac{4}{1.046635} = 3.8218.$$

Thus the answer is 3.82.

Question 15.

An insurer has a liability that is expected to result in the following cash outflows.

End of year	Cash outflow
1	10
2	12
3	15
4	20
5	30

The insurer uses an 8% annual effective interest rate to discount future cash flows. Calculate the Macaulay duration of this liability.

(A) 3.1 years

(B) 3.2 years

(C) 3.4 years

(D) 3.5 years

(E) 3.6 years

Solution.

The Macaulay duration is

$$D_{\text{mac}} = \frac{\sum tC_t(1.08)^{-t}}{\sum C_t(1.08)^{-t}}.$$

The denominator is

$$\frac{10}{1.08} + \frac{12}{1.08^2} + \frac{15}{1.08^3} + \frac{20}{1.08^4} + \frac{30}{1.08^5} = 66.58.$$

The numerator is

$$\frac{1(10)}{1.08} + \frac{2(12)}{1.08^2} + \frac{3(15)}{1.08^3} + \frac{4(20)}{1.08^4} + \frac{5(30)}{1.08^5} = 226.47.$$

Thus

$$D_{\text{mac}} = \frac{226.47}{66.58} = 3.4015.$$

The answer is 3.4 years.

Question 16.

A firm has a liability cash flow of 100 at the end of year two and a second liability cash flow of 200 at the end of year three.

The firm also has asset cash flows of X at the end of years one and five.

Using an annual effective interest rate of 10%, calculate the absolute value of the difference between the Macaulay durations of the asset and liability cash flows.

(A) 0.018

(B) 0.020

(C) 0.022

(D) 0.024

(E) 0.026

Solution.

Let $v = 1/1.10$. The liability duration is

$$D_L = \frac{2(100v^2) + 3(200v^3)}{100v^2 + 200v^3} = 2.6452.$$

The asset duration is

$$D_A = \frac{1(Xv) + 5(Xv^5)}{Xv + Xv^5} = \frac{v + 5v^5}{v + v^5} = 2.6233.$$

Therefore

$$|D_A - D_L| = |2.6233 - 2.6452| = 0.0219.$$

The answer is 0.022.

Question 17.

A company created a portfolio in order to protect its position using Redington immunization.

Under which of the following changes in the yield rate is the immunization strategy guaranteed to be effective?

(A) Only a small decrease in the yield rate

(B) Only a small increase in the yield rate

(C) Only a small change in the yield rate

(D) Any decrease in the yield rate

(E) Any change in the yield rate

Solution.

Redington immunization is a local immunization method. It matches present value and duration and requires asset convexity to exceed liability convexity. Since the method is based on a local Taylor approximation, it is guaranteed only for sufficiently small changes in the yield rate.

Thus the correct answer is: only a small change in the yield rate.

Question 18.

A company has liabilities of 1000 and 300 due at the end of years two and four, respectively. The company develops an investment program that produces asset cash flows of X and Y at the end of years one and three, respectively. The investment portfolio is constructed to match the present value and duration of the company's payment obligations based on an annual effective rate of interest of 5%.

Calculate Y/X.

(A) 2.14

(B) 2.75

(C) 3.42

(D) 4.05

(E) 4.91

Solution.

Let $v = 1/1.05$. Matching present values gives

$$Xv + Yv^3 = 1000v^2 + 300v^4. \tag{1}$$

Matching duration is equivalent to matching the present-value-weighted time sums:

$$Xv + 3Yv^3 = 2(1000v^2) + 4(300v^4). \tag{2}$$

Subtracting (1) from (2),

$$2Yv^3 = 1000v^2 + 900v^4,$$

so

$$Y = 953.57.$$

Substituting in (1) gives

$$X = 346.61.$$

Therefore

$$\frac{Y}{X} = \frac{953.57}{346.61} = 2.75.$$

Question 19.

A 10,000 par value 10-year bond with 8% annual coupons is bought at a premium to yield an annual effective rate of 6%.

Calculate the interest portion of the 7th coupon.

(A) 632

(B) 642

(C) 651

(D) 660

(E) 667

Solution.

The book value at time 6 is the present value of future payments:

$$BV_6 = 10{,}000v^4 + 800a_{\overline{4}|0.06} = 7920.94 + 2772.08 = 10{,}693.$$

The interest portion is

$$10{,}693(0.06) = 641.58.$$

Thus the answer is 642.

Question 20.

Three asset-liability cash flows are given in the following table, where a positive amount is an asset cash flow and a negative amount is a liability due at the corresponding time.

t in years	0	1	2	3
X	102,400	-192,000	0	100,000
Y	158,400	-342,000	100,000	100,000
Z	-89,600	288,000	100,000	-300,000

Determine which set of cash flows is Redington immunized for an annual effective interest rate of $i = 25\%$.

(A) X only

(B) Y only

(C) Z only

(D) X, Y, and Z

(E) The correct answer is not given by (A), (B), (C), or (D).

Solution.

Let $h(i)$ be the present value of the cash flows. For Redington immunization, the value of the function and its first derivative at 25% must be zero, and the second derivative must be positive.

X is immunized because

$$h(0.25) = 102{,}400 - \frac{192{,}000}{1.25} + \frac{100{,}000}{1.25^3} = 0,$$

$$h'(0.25) = \frac{192{,}000}{1.25^2} - \frac{100{,}000(3)}{1.25^4} = 0,$$

and

$$h''(0.25) = -\frac{192{,}000(2)}{1.25^3} + \frac{100{,}000(3)(4)}{1.25^5} = 196{,}608 > 0.$$

Y is not immunized because

$$h'(0.25) = \frac{342{,}000}{1.25^2} - \frac{100{,}000(2)}{1.25^3} - \frac{100{,}000(3)}{1.25^4} = -6400 \neq 0.$$

Z is not immunized because

$$h(0.25) = -89{,}600 + \frac{288{,}000}{1.25} + \frac{100{,}000}{1.25^2} - \frac{300{,}000}{1.25^3} = 51{,}200 \neq 0.$$

Thus the answer is X only.