Dept of Mathematics and Statistics
King Fahd University of Petroleum & Minerals

AS481: Actuarial Contingencies 2

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Major Exam I, Term 252
Monday, February 23, 2026
2:00 PM – 4:00 PM

Name………………………………………………… ID#:_________ Serial #:___

Instructions.

1. Any student caught with mobile phones on during the exam will be considered under the cheating rules of the University.
2. If you need to leave the room, please do so before the exam starts. Nobody will be allowed to leave the room once the exam starts.
3. Only materials provided by the instructor can be present on the table during the exam.
4. Do not spend too much time on any one question. If a question seems too difficult, leave it and go on.
5. Use the blank portions of each page for your work. Extra blank pages can be provided if necessary. If you use an extra page, indicate clearly what problem you are working on.
6. ***Only answers supported by work will be considered. Unsupported guesses will not be graded.***
7. While every attempt is made to avoid defective questions, sometimes they do occur. In the rare event that you believe a question is defective, the instructor cannot give you any guidance beyond these instructions.
8. Mobile calculators, I-pad, or communicable devices are disallowed. Use regular scientific calculators, financial calculators, or SOA approved calculators only. ***Write important steps to arrive at the solution of the exam problems.***

The test is 120 minutes, GOOD LUCK, and you may begin now!

Question	Total Marks	Marks Obtained	Comments
1-5	5		
6-10	5		
PART B			
1	3		
2	4		
Bonus	2		
Total	17		

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PART A

1. Consider a discrete-time Markov chain with state space $\{1, 2, 3, 4\}$.
Transitions are as follows:

- From state 1, the chain moves to state 2 with probability 0.6 or state 3 with probability 0.4.

- From state 2, it returns to state 1 with probability 0.3 or moves to state 3 with probability 0.7.

- From state 3, it remains in state 3 with probability 1.

- State 4 has no outgoing transitions (i.e., once entered, the chain stays in 4).

Which statement correctly identifies the nature of states 1, 2, 3, and 4?

A. State 1 is absorbing, state 2 is transient, state 3 is recurrent, and state 4 is transient.

B. State 1 is recurrent, state 2 is recurrent, state 3 is absorbing, and state 4 is transient.

C. State 1 is transient, state 2 is transient, state 3 is absorbing, and state 4 is absorbing.

D. State 1 is transient, state 2 is recurrent, state 3 is transient, and state 4 is absorbing.

E. State 1 is recurrent, state 2 is transient, state 3 is transient, and state 4 is absorbing.

2. A Markov process models the health status of an insured over time. The transition probability from state "Healthy" (H) to "Disabled" (D) satisfies:

$$P(X_{t+1} = D \mid X_t = H) = \begin{cases} 0.02, & 0 \leq t < 40, \\ 0.05, & t \geq 40. \end{cases}$$

Additionally, transitions between all other pairs of states have probabilities that do not depend on time. Which statement is correct?

A. The process is homogeneous because only one transition probability depends on time.

B. The process is homogeneous because every state eventually transitions to a time-independent absorbing state.

C. The process is non-homogeneous because at least one transition probability varies with t.

D. The process is homogeneous since transition probabilities depend only on the current state, not on past states.

E. The process is non-homogeneous only if *all* transitions depend on t.

3. Which of the following is an example of a competing risks scenario in actuarial science?

A) A pension plan where a member may retire, die, or leave for another job, but only one of these events will occur
B) A policyholder simultaneously holding multiple insurance policies
C) A life insurance policy that covers accidental death and natural death, paying out for both events
D) A medical plan that provides coverage for two different treatments.
E) All of the above.

4. In a discrete-time Markov chain with state space $\{1,2,3,\ldots,m\}$, let

$\pi_n = (\pi_{1n}, \pi_{2n}, \ldots, \pi_{mn})$. Which of the statements is correct?

A. π_n is the transition matrix at time n, and π_{in} is the probability of moving from state i to state n.

B. π_n is the distribution of *first passage times*, and π_{in} is the probability that the process first enters state i at time n.

C. π_n is the vector of state-specific expected rewards at time n, and π_{in} gives the expected reward of being in state i at time n.

D. π_n is the state probability vector at time n, and π_{in} is the probability that the chain is in state i at time n.
E. π_n is the vector of stationary probabilities, and π_{in} is the long-run proportion of transitions into state i over all time.

5. A certain animal species can be classified as thriving (State 0), endangered (State 1), or extinct (State 2). Movement among states is governed by a non-homogeneous Markov process defined by the following transition probability matrices:

$$\mathbf{P}^{(0)} = \begin{bmatrix} .85 & .15 & 0 \\ 0 & .70 & .30 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{P}^{(1)} = \begin{bmatrix} .90 & .10 & 0 \\ .10 & .70 & .20 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{P}^{(2)} = \begin{bmatrix} .95 & .05 & 0 \\ .20 & .70 & .10 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{P}^{(k)} = \begin{bmatrix} .95 & .05 & 0 \\ .50 & .50 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $k = 3, 4, 5, \cdots$.
If the species is thriving at time n, is it possible for it to be extinct at time n +1?

A. Yes
B. No
C. Not enough information to decide
D. It's confusing
E. None of the above

Show workings for Questions 6 – 10

6. A discrete-time Markov process has only two states, denoted as State 0 and State 1. A person is age x at time 0, and is located in State 0. We are given the age-specific one-year transition probability values

$$p_{x+k}^{00} = .70 + \frac{.10}{k+1}$$

and

$$p_{x+k}^{11} = .60 + \frac{.20}{k+1}.$$

Find the value of p_{x+2}^{10}.

A. 0.6333
B. 0.2500
C. 0.3333
D. 0.3145
E. 0.7500

7.

If $\mu_{x+t}^{(1)} = .10$ and $\mu_{x+t}^{(2)} = .20$ for all t, find

${}_{\infty}q_x^{(1)}$

A. 0.3333
B. 0.6667
C. 0.2000
D. 0.8000

Use the following information to answer questions 8-10.

Given the following probabilities of decrement. Assume an initial group (called the radix of the table) of size 1000. Calculate

x	$q_x^{(1)}$	$q_x^{(2)}$
45	.011	.100
46	.012	.100
47	.013	.100
48	.014	.100
49	.015	.100
50	.016	.100

8. ${}_3p_{46}^{\tau}$

A. 0.6979
B. 0.6204
C. 0.8890
D. 0.3145
E. 0.0235

9. ${}_2d_{47}^2$

A. 78.94
B. 70.02
C. 148.96
D. 62.04
E. 20.06

10. ${}_2q_{46}^1$

A. 0.6979
B. 0.6204
C. 0.8890
D. 0.3145
E. 0.0235

PART B

1. Solve the Kolmogorov differential equation for
${}_np_x^{00}$
,
and translate the result into standard actuarial notation.

2.

Students can leave a certain three-year school only for reasons of failure (Decrement 1) or voluntary withdrawal (Decrement 2), where each decrement is uniformly distributed over $(x, x+1)$ in its associated single-decrement table. The following values are given:

x	$q_x^{\prime(1)}$	$q_x^{\prime(2)}$	$q_x^{(1)}$	$q_x^{(2)}$
0	.10	.25	–	–
1	.20	.20	–	–
2	.20	.10	–	–

(a) Given that a person decrements from school in the third year, find the probability that the decrement was a failure.

(b) Given that a student enters Year 2, find the probability of eventually decrementing due to failure.

BONUS

The following double-decrement table gives probability values for a student at the beginning of each year in a three-year Actuarial Science Graduate School. Some of the entries in the table have been obliterated by tear stains.

Academic Year	Probability of Academic Failure	Probability of Voluntary Withdrawal	Probability of Completing the Year
1	.40	.20	–
2	–	.30	–
3	–	–	.60

It is known that ten times as many students complete Year 2 as fail during Year 3, and the number of students who fail during Year 2 is 40% of the number who complete Year 2. Find the probability that a new student entering the school will voluntarily withdraw before graduation.

Chapter 3 MQR Markov Chain review

$$p^{ij} = P\left(X_{n+1} = j | X_n = i\right) \qquad \sum_{j=0}^{m} p^{ij} = 1 \qquad \mathbf{P} = \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0m} \\ p_{10} & p_{11} & \cdots & p_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m0} & p_{m1} & \cdots & p_{mm} \end{bmatrix}$$

$$\boldsymbol{\pi}_n = \left(\pi_{0n}, \pi_{1n}, \cdots, \pi_{mn}\right) \qquad \sum_{i=0}^{m} \pi_{in} = 1 \qquad \boldsymbol{\pi}_{n+r} = \begin{cases} \boldsymbol{\pi}_n \cdot \mathbf{P} \cdot \mathbf{P} \cdot \cdots \cdot \mathbf{P} = \boldsymbol{\pi}_n \cdot \mathbf{P}^r & \text{homogeneous} \\ \boldsymbol{\pi}_n \cdot \mathbf{P}^{(0)} \cdot \mathbf{P}^{(1)} \cdot \cdots \cdot \mathbf{P}^{(r-1)} & \text{non-homogeneous} \end{cases}$$

$${}_r p_n^{ij} = P\left(X_{n+r} = j | X_n = i\right) \qquad {}_r p_n^{\overline{ii}} \leq {}_r p_n^{ii}$$

$${}_r p_n^{\overline{ii}} = {}_r p_n^{ii} \quad \text{if } \textbf{cannot} \text{ reenter state } i \text{ once left } i$$

μ_{x+s}^{ij} =force of transition. $\quad \mu_{x+s}^{i} = \sum_{j=1}^{i-1} \mu_{x+s}^{ij} + \sum_{j=i+1}^{m} \mu_{x+s}^{ij}$ =force of transitioning **out**.

$$\frac{d}{dr}\left({}_r p_x^{ij}\right) = \sum_{k \neq j} \left({}_r p_x^{ik} \mu_{x+r}^{kj} - {}_r p_x^{ij} \cdot \mu_{x+r}^{jk}\right) = \sum_{k \neq j} \left({}_r p_x^{ik} \mu_{x+r}^{kj}\right) - {}_r p_{ij}^{(t)} \cdot \mu_{x+r}^{i}$$

Chapter 13 MQR Multiple Decrement Models: Theory

OBJECTIVES: 1.To understand the concept of a multiple decrement table

2.To understand the force of decrement

3.To construct a multiple decrement model using associated single decrements and to apply various assumptions to calculate rates for discrete jumps.

13.1 Discrete Multiple Decrement Models

$q_x^{(\tau)} = q_x^{(1)} + q_x^{(2)} + \ldots + q_x^{(m)} = \sum_{j=1}^{m} q_x^{(j)}$ (13.1) $\qquad p_x^{(\tau)} = 1 - q_x^{(\tau)}$ (13.2)

$d_x^{(\tau)} = \sum_{j=1}^{m} d_x^{(j)} = \ell_x^{(\tau)} \cdot q_x^{(\tau)}$ (13.3 & 13.7b) $\qquad {}_n p_x^{(\tau)} = 1 - {}_n q_x^{(\tau)}$ (13.7e)

${}_n d_x^{(j)} = \sum_{t=0}^{n-1} d_{x+t}^{(j)} = \ell_x^{(\tau)} \cdot {}_n q_x^{(j)}$ (13.4 & 13.7c) $\qquad \ell_x^{(\tau)} = \sum_{j=1}^{m} \ell_x^{(j)}$ (13.6)

$d_x^{(j)} = \ell_x^{(\tau)} \cdot q_x^{(j)}$ (13.7a) $\qquad \ell_{x+n}^{(\tau)} = \ell_x^{(\tau)} \cdot {}_n p_x^{(\tau)}$ (13.7f)

${}_n d_x^{(\tau)} = \sum_{j=1}^{m} {}_n d_x^{(j)} = \ell_x^{(\tau)} \cdot {}_n q_x^{(\tau)}$ (13.5a & 13.7d) $\qquad {}_n q_x^{(\tau)} = {}_n d_x^{(\tau)} / \ell_x^{(\tau)} = \sum_{j=1}^{m} {}_n q_x^{(j)}$ (13.5b)

13.1.2 Random Variable Analysis

The *joint probability function* of K_x and J_x is $\Pr\left(K_x = k \cap J_x = j\right) = {}_{k|} q_x^{(j)} = \dfrac{d_{x+k}^{(j)}}{\ell_x^{(\tau)}}$ (13.8)

The *marginal probability functions* are

i) $\Pr\left(K_x = k\right) = \sum_{j=1}^{m} \Pr\left(K_x = k \cap J_x = j\right) = {}_{k|} q_x^{(\tau)} = \dfrac{d_{x+k-1}^{(1)} + \ldots + d_{x+k}^{(m)}}{\ell_x^{(\tau)}} = \sum_{j=1}^{m} \dfrac{d_{x+k}^{(j)}}{\ell_x^{(\tau)}}$ (13.9)

ii) $\Pr\left(J_x = j\right) = \sum_{k=1}^{\infty} \Pr\left(K_x = k \cap J_x = j\right) = \sum_{k=1}^{\infty} \dfrac{d_{x+k}^{(j)}}{\ell_x^{(\tau)}}$ (13.10)

13.2 Theory of Competing Risks

${}_n q_x^{\prime(j)} \geq {}_n q_x^{(j)}$ (13.11)

13.3 Continuous Multiple Decrement Models

$\mu_{x+t}^{(j)} = \dfrac{-\frac{d}{dt}\ {}_t p_x^{\prime(j)}}{{}_t p_x^{\prime(j)}}$ (13.12a) $\qquad \mu_{x+t}^{(\tau)} = \dfrac{-\frac{d}{dt}\ {}_t p_x^{(\tau)}}{{}_t p_x^{(\tau)}}$ (13.13a)

${}_t q_x^{\prime(j)} = \int_0^t {}_s p_x^{\prime(j)} \mu_{x+s}^{(j)}\ ds = 1 - {}_t p_x^{\prime(j)}$ $\qquad {}_t p_x^{\prime(j)} = \exp\left(-\int_0^t \mu_{x+s}^{(j)}\ ds\right)$ (13.12b)

$f_{x^{(j)}}(t) = {}_t p_x^{\prime(j)} \cdot \mu_{x+t}^{(j)}$ (13.14) $\qquad {}_t p_x^{(\tau)} = \exp\left(-\int_0^t \mu_{x+s}^{(\tau)}\ ds\right)$ (13.13b)

$F_{x^{(j)}}(t) = \Pr\left[T_x^{(j)} \leq t\right] = \int_0^t f_{x^{(j)}}(s)\ ds = \int_0^t {}_s p_x^{\prime(j)} \cdot \mu_{x+s}^{(j)}\ ds$ (13.15)

$$\begin{aligned} \mu_{x+t}^{(\tau)} &= \frac{-\frac{d}{dt}\ {}_t p_x^{(\tau)}}{{}_t p_x^{(\tau)}} = -\frac{d}{dt} \ln_t p_x^{(\tau)} = -\frac{d}{dt} \ln\left[{}_t p_x^{\prime(1)} \cdot {}_t p_x^{\prime(2)} \cdot \ldots \cdot {}_t p_x^{\prime(m)}\right] \\ &= \left(-\frac{d}{dt} \ln\ {}_t p_x^{\prime(1)}\right) + \left(-\frac{d}{dt} \ln\ {}_t p_x^{\prime(2)}\right) + \ldots + \left(-\frac{d}{dt} \ln\ {}_t p_x^{\prime(m)}\right) = \mu_{x+t}^{(1)} + \mu_{x+t}^{(2)} + \ldots + \mu_{x+t}^{(m)} = \sum_{j=1}^{m} \mu_{x+t}^{(j)} \end{aligned} \qquad (13.17)$$

Fundamental Relation Between Primed and Unprimed Rates: $_tp_x^{(\tau)} = \exp\left(-\sum_{j=1}^{m}\int_0^t \mu_{x+s}^{(j)}\ ds\right) = \prod_{j=1}^{m} {}_tp_x'^{(j)}$ (13.16)

$_tq_x^{(j)} = \int_0^t f_{T,J}(s,j)\ ds = \int_0^t {}_sp_x^{(\tau)} \cdot \mu_{x+s}^{(j)}\ ds$ (13.18) $\quad$ $_tq_x^{(\tau)} = \int_0^t {}_sp_x^{(\tau)} \cdot \mu_{x+s}^{(\tau)}\ ds$ (13.20)

$\frac{d}{dt}\ {}_tq_x^{(j)} = \frac{d}{dt}\int_0^t {}_sp_x^{(\tau)} \cdot \mu_{x+s}^{(j)}\ ds = {}_tp_x^{(\tau)} \cdot \mu_{x+s}^{(j)} \ \rightarrow\ \mu_{x+t}^{(j)} = \frac{\frac{d}{dt}\ {}_tq_x^{(j)}}{{}_tp_x^{(\tau)}}$ (13.19)

Joint Distribution of T_x and J_x $\quad$ $\Pr(t < T_x \le t + dt \text{ and } J_x = j) \approx {}_tp_x^{(\tau)}\mu_{x+t}^{(j)}dt,$ $\quad$ $_tq_x^{(j)} = \int_0^t {}_sp_x^{(\tau)}\ \mu_{x+s}^{(j)}\ ds.$

13.4.1 Uniform Distribution of Decrements in the Multiple Decrement Context

$_tq_x^{(j)} = t \cdot q_x^{(j)}$ (13.21) $\quad$ $q_x^{(j)} = {}_tp_x^{(\tau)} \cdot \mu_{x+t}^{(j)}$ (13.22)

$_tq_x^{(\tau)} = t \cdot q_x^{(\tau)}$ (13.23) $\quad$ $_tp_x^{(\tau)} = 1 - t \cdot q_x^{(\tau)}$ (13.24)

$\mu_{x+t}^{(j)} = \frac{q_x^{(j)}}{{}_tp_x^{(\tau)}} = \frac{q_x^{(j)}}{1 - t \cdot q_x^{(\tau)}}$ (13.25) $\quad$ $_tp_x'^{(j)} = \exp\left[\frac{q_x^{(j)}}{q_x^{(\tau)}} \cdot \ln\left(1 - t \cdot q_x^{(\tau)}\right)\right] = \left(1 - t \cdot q_x^{(\tau)}\right)^{q_x^{(j)}/q_x^{(\tau)}}$ (13.26)

13.4.2 Uniform Distribution in the Associated Single- Decrement Tables

case	$_tq_x'^{(j)} = t \cdot q_x'^{(j)}$	(13.27)	$_tp_x'^{(j)} \cdot \mu_{x+t}^{(j)} = q_x'^{(j)}$	(13.28)
Double decrement	$q_x^{(1)} = \int_0^1 \left(1 - t \cdot q_x'^{(2)}\right) \cdot q_x'^{(1)}dt = q_x'^{(1)}\left(1 - \frac{1}{2} \cdot q_x'^{(2)}\right)$	(13.29a)	$q_x^{(2)} = q_x'^{(2)}\left(1 - \frac{1}{2} \cdot q_x'^{(1)}\right)$	(13.29b)
Triple Decrement	$q_x^{(1)} = q_x'^{(1)}\left[1 - \frac{1}{2}\left(q_x'^{(2)} + q_x'^{(3)}\right) + \frac{1}{3}\left(q_x'^{(2)} \cdot q_x'^{(3)}\right)\right]$	(13.30)		

13.4.3 Constant Force of Decrement

$\frac{{}_tq_x^{(j)}}{{}_tq_x^{(\tau)}} = \frac{\mu_{x+t}^{(j)}}{\mu_{x+t}^{(\tau)}}$

Miscellaneous Results (From ACTEX MLC manual)

1. Assumptions on the single decrement table.

Backing out the Unprimed Rates from Primed Rates

$$_sq_x^{(i)} = \int_0^s {}_tp_x^{(\tau)}\mu_{x+t}^{(i)}\ dt = \int_0^s \left[\prod_{j=1,\ j\neq i}^{m} {}_tp_x'^{(j)}\right] {}_tp_x'^{(i)}\mu_{x+t}^{(i)}\ dt$$

$$\frac{d}{dr}({}_rp_x^{ij}) = \sum_{k\neq j}\left({}_rp_x^{ik}\mu_{x+r}^{kj} - {}_rp_x^{ij} \cdot \mu_{x+r}^{jk}\right) = \sum_{k\neq j}\left({}_rp_x^{ik}\mu_{x+r}^{kj}\right) - {}_rp_{ij}^{(t)} \cdot \mu_{x+r}^{i}$$

2. **Constant Force Assumption for Multiple Decrements** $\quad$ For any $t \in [0,1]$ and integer-valued x,

(i) $_tp_x^{(\tau)} = \left[p_x^{(\tau)}\right]^t$ $\quad$ (survival probability for **fractional** ages)

(ii) *Ratio* Property : $\frac{{}_tq_x^{(i)}}{{}_tq_x^{(\tau)}} = \frac{\mu_{x+s}^{(i)}}{\mu_{x+s}^{(\tau)}}$ for any $s \in [0,1]$ $\quad$ (To get unprimed rates from (i))

(iii) *Partition* Property : $_tp_x^{t(i)} = \left[{}_tp_x^{(\tau)}\right]^{q_x^{(i)}/q_x^{(\tau)}}$ $\quad$ (To get primed rates from unprimed rates from (i))

3. **Uniform Distribution of Death (UDD) for Multiple Decrement (MUDD) Table**

For any $t \in [0,1]$ and integer-valued x,

(i) $_tp_x^{(\tau)}\mu_{x+t}^{(i)} = q_x^{(i)}$ or equivalently $\mu_{x+t}^{(i)} = \frac{q_x^{(i)}}{1 - tq_x^{(\tau)}}$for $t \neq 1$

(ii) *Ratio* Property : $\frac{{}_tq_x^{(i)}}{{}_tq_x^{(\tau)}} = \frac{\mu_{x+s}^{(i)}}{\mu_{x+s}^{(\tau)}}$ for any $s \in [0,1]$

(iii) *Partition* Property : $_tp_x'^{(i)} = \left[{}_tp_x^{(\tau)}\right]^{q_x^{(i)}/q_x^{(\tau)}}$ (To get primed rates from unprimed rates $_tq_x^{(i)}$ and $_tp_x^{(\tau)}$)

Discrete jumps: Handling Both Discrete and Continuous Decrement

1) $_sq_x^{(i)} = \int_0^s \left[\prod_{j=1,\ j\neq i}^{m} {}_tp_x'^{(j)}\right] {}_tp_x'^{(i)}\mu_{x+t}^{(i)}dt$ $\quad$ holds when decrement i is **continuous.**

2) $_sq_x^{(i)} = \sum_{t_k \le s}\left[\prod_{j=1,\ j\neq i}^{m} {}_{t_k}p_x'^{(j)}\right]\Delta({}_{t_k}q_x'^{(i)})$ $\quad$ holds when decrement i is **discrete**

Here t_k are the jump times and $\Delta\left({}_{t_k}q_x'^{(i)}\right)$ is the jump size at time t_k.

Case	Given	Sought	Assumption	Results
A1	$q_x'^{(1)}$ and $q_x'^{(2)}$	${}_tq_x^{(1)}$ and ${}_tq_x^{(2)}$	SUDD	${}_tq_x^{(1)} = q_x'^{(1)}\left(t - \frac{t^2}{2}\cdot q_x'^{(2)}\right)$ ${}_tq_x^{(2)} = q_x'^{(2)}\left(t - \frac{t^2}{2}\cdot q_x'^{(1)}\right)$
A2	$q_x'^{(1)}$ and $q_x'^{(2)}$	$q_x^{(1)}$ and $q_x^{(2)}$	SUDD	$q_x^{(1)} = q_x'^{(1)}\left(1 - \frac{1}{2}\cdot q_x'^{(2)}\right)$ $q_x^{(2)} = q_x'^{(2)}\left(1 - \frac{1}{2}\cdot q_x'^{(1)}\right)$
B1	$q_x'^{(1)}$ and $q_x'^{(2)}$	${}_tq_x^{(1)}$ and ${}_tq_x^{(2)}$	CF	${}_tq_x^{(1)} = \frac{\mu^{(1)}}{\mu^{(\tau)}}\cdot\left(1 - {}_tp_x^{(\tau)}\right)$ ${}_tq_x^{(2)} = \frac{\mu^{(2)}}{\mu^{(\tau)}}\cdot\left(1 - {}_tp_x^{(\tau)}\right)$
B2	$q_x'^{(1)}$ and $q_x'^{(2)}$	$q_x^{(1)}$ and $q_x^{(2)}$	CF	$q_x^{(1)} = \frac{\mu^{(1)}}{\mu^{(\tau)}}\cdot q_x^{(\tau)}$ $q_x^{(2)} = \frac{\mu^{(2)}}{\mu^{(\tau)}}\cdot q_x^{(\tau)}$
C1	$q_x^{(1)}$ and $q_x^{(2)}$	${}_tq_x'^{(1)}$ and ${}_tq_x'^{(2)}$	MUDD	${}_tq_x'^{(1)} = 1 - \left(1 - t\cdot q_x^{(\tau)}\right)^{q_x^{(1)}/q_x^{(\tau)}}$ ${}_tq_x'^{(2)} = 1 - \left(1 - t\cdot q_x^{(\tau)}\right)^{q_x^{(2)}/q_x^{(\tau)}}$
C2	$q_x^{(1)}$ and $q_x^{(2)}$	$q_x'^{(1)}$ and $q_x'^{(2)}$	MUDD	$q_x'^{(1)} = 1 - \left(1 - q_x^{(\tau)}\right)^{q_x^{(1)}/q_x^{(\tau)}}$ $q_x'^{(2)} = 1 - \left(1 - q_x^{(\tau)}\right)^{q_x^{(2)}/q_x^{(\tau)}}$
D1	$q_x^{(1)}$ and $q_x^{(2)}$	${}_tq_x'^{(1)}$ and ${}_tq_x'^{(2)}$	CF	${}_tq_x'^{(1)} = 1 - \left({}_tp_x^{(\tau)}\right)^{{}_tq_x^{(1)}/{}_tq_x^{(\tau)}}$ ${}_tq_x'^{(2)} = 1 - \left({}_tp_x^{(\tau)}\right)^{{}_tq_x^{(2)}/{}_tq_x^{(\tau)}}$
D2	$q_x^{(1)}$ and $q_x^{(2)}$	$q_x'^{(1)}$ and $q_x'^{(2)}$	CF	$q_x'^{(1)} = 1 - \left(p_x^{(\tau)}\right)^{q_x^{(1)}/q_x^{(\tau)}}$ $q_x'^{(2)} = 1 - \left(p_x^{(\tau)}\right)^{q_x^{(2)}/q_x^{(\tau)}}$
E	$q_x'^{(1)}$ and $q_x'^{(2)}$	$q_x^{(1)}$ and $q_x^{(2)}$	MUDD	$q_x^{(1)} = q_x^{(\tau)}\cdot\left[\frac{\ln p_x'^{(1)}}{\ln p_x^{(\tau)}}\right]$ $q_x^{(2)} = q_x^{(\tau)}\cdot\left[\frac{\ln p_x'^{(2)}}{\ln p_x^{(\tau)}}\right]$
F	$q_x^{(1)}$ and $q_x^{(2)}$	$q_x'^{(1)}$ and $q_x'^{(2)}$	SUDD	See Solution