King Fahd University of Petroleum and Minerals
College of Computing and Mathematics
Department of Mathematics

Math 323 (Term 252)

Major Exam 1 (Two Hours)

Problem 1. Prove the following assertions:

(**1**) The groups $\mathbb{Q}$ and $\mathbb{Q}^\star$ are not isomorphic.
(**2**) Let G be an abelian group. Then, $H := \Big\{x \in G \mid \big|x\big| \text{ is finite}\Big\}$ is a subgroup of G.
(**3**) Let G be a group which has only two subgroups $\Big\{1\Big\}$ and G. Then, G is cyclic of prime order.

Problem 2. Let S_{12} denote the symmetric group of degree 12 and let $\sigma \in S_{12}$ with $\sigma = \alpha_1\alpha_2\cdots\alpha_k$ where the α_i's are disjoint r_i-cycles such that $r_i \geq 3$, for each i, and $r_1 + r_2 + \cdots + r_k = 12$. Find all possible cycle decompositions of σ and, for each case, determine the order of σ and whether σ is an odd or even permutation.

Problem 3. .

(**1**) Determine all possible generators of the cyclic group $\dfrac{\mathbb{Z}}{70\mathbb{Z}}$.

(**2**) Draw the lattice of all its subgroups (i.e., with all possible inclusions between subgroups and, for each subgroup, give its generator and order).

Problem 4. Let $G = U(39)$ be the multiplication group modulo 39.

(**1**) Find the order of 2.
(**2**) Find two elements of order 2.
(**3**) Determine the isomorphism class of G.
(**4**) Express G as an internal direct product of two cyclic subgroups.

Problem 5. Let G be a finite group. Let H be a normal subgroup of G and let

$$\widehat{H} := \Big\{g \in G \mid ghg^{-1} = h \ , \ \forall\, h \in H\Big\}.$$

(**1**) Show that $\widehat{H}$ is a normal subgroup of G.

(**2**) Let $g \in G$. Show that the mapping $\begin{array}{rccc} \phi_g : & H & \longrightarrow & H \\ & h & \longmapsto & ghg^{-1} \end{array}$ is an automorphism of H.

(**3**) Show that the mapping $\begin{array}{rccc} \phi : & G & \longrightarrow & \mathrm{Aut}(H) \\ & g & \longmapsto & \phi_g \end{array}$ is a group homomorphism and find its kernel.

(**4**) Show that the order of the factor group $\dfrac{G}{\widehat{H}}$ divides the order of $\mathrm{Aut}(H)$.

(**5**) Assume $\big|G\big| = pq$ and $\big|H\big| = p$, where $p \lneqq q$ are prime numbers. Show that $\Big|\mathrm{Aut}(H)\Big| = p - 1$.

(**6**) Prove that $\dfrac{G}{Z(G)}$ is cyclic.