King Fahd University of Petroleum and Minerals
College of Computing and Mathematics
Department of Mathematics

Math 323 (Term 252)

Major Exam 2 (Two hours)

Problem 1. Find the remainder when X^{48} is divided by $X+10$ in $\frac{\mathbb{Z}}{13\mathbb{Z}}\left[X\right]$.

Problem 2. Find the number of **distinct solutions** for the equation $x^{60}=1$ in the field $\frac{\mathbb{Z}}{29\mathbb{Z}}$.

Problem 3. Let n be a positive integer with decimal representation $a_k a_{k-1}\cdots a_1 a_0$.

(1) Show that n is divisible by 11 if and only if the alternating sum of its digits is divisible by 11.
(2) Find the smallest 10-digit number that is divisible by 11.
(3) Show that n is divisible by 8 if and only if the number formed by its last 3 digits is divisible by 8.
(4) Find the greatest 10-digit number that is divisible by 8.

Problem 4. Consider the following rings given by

$$A := \left\{\begin{pmatrix} n & m \\ -m & n \end{pmatrix} \mid n,m \in \mathbb{Z}\right\} \;;\; B := \left\{\begin{pmatrix} n & m \\ 0 & n \end{pmatrix} \mid n,m \in \mathbb{Z}\right\} \;;\; C := \left\{\begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \mid a,b \in \frac{\mathbb{Z}}{13\mathbb{Z}}\right\}$$

(1) Show that A is isomorphic to the Gaussian ring $\mathbb{Z}[i]$.
(2) Show that B is NOT isomorphic to the ring $\mathbb{Z}\times\mathbb{Z}$.
(3) Find an ideal of $\mathbb{Z}[X]$ such that the quotient ring by this ideal is isomorphic to B (justify).
(4) Find the number of all nilpotent elements of the ring C.

Problem 5. Let R be a commutative ring. Let I,J be two **co-maximal** ideals of R and consider the **ring homomorphism**

$$\begin{array}{rccc} \phi: & R & \longrightarrow & \frac{R}{I}\times\frac{R}{J} \\ & r & \longmapsto & (\,\bar{r}\,,\,\bar{r}\,) \end{array}$$

(1) Show that $IJ = I\cap J$.
(2) Find $\mathrm{Ker}(\phi)$.
(3) Show that ϕ is onto.
(4) Show that $\frac{R}{IJ}\cong\frac{R}{I}\times\frac{R}{J}$ as rings.
(5) Deduce that if $m,n\in\mathbb{N}$ with $(m,n)=1$, then $\frac{\mathbb{Z}}{n\mathbb{Z}}\times\frac{\mathbb{Z}}{m\mathbb{Z}}\cong\frac{\mathbb{Z}}{nm\mathbb{Z}}$ as rings.
(6) Consider the ring $A := \frac{\mathbb{Z}}{2\mathbb{Z}}\times\frac{\mathbb{Z}}{3\mathbb{Z}}$ and its **principal** ideal $I := (\bar{1},\bar{0})A$. Use **two different methods** to find $\left|\frac{A}{I}\right|$.