King Fahd University of Petroleum and Minerals
College of Computing and Mathematics
Department of Mathematics

Math 323 (Term 252)

Final Exam (150 minutes - 20 Exercises)

CODE 1

Exercise 1. Let

$$H := \left\{x \in \mathbb{R}^\star \mid x = 1 \text{ or } x \text{ irrational}\right\} \quad ; \quad K := \left\{x \in \mathbb{R}^\star \mid x \geq 1\right\} \quad ; \quad L := \left\{A \in M_n(\mathbb{R}) \mid \det(A) = 1\right\}$$

Then, which one of the following statements is CORRECT?

(a) H is a subgroup of $\mathbb{R}^\star$, and K is not a subgroup of $\mathbb{R}^\star$

(b) H is a subgroup of $\mathbb{R}^\star$, and L is a subgroup of $GL(n, \mathbb{R})$

(c) K is not a subgroup of $\mathbb{R}^\star$, and L is not a subgroup of $GL(n, \mathbb{R})$

(d) K is not a subgroup of $\mathbb{R}^\star$, and L is a subgroup of $GL(n, \mathbb{R})$

(e) None of the above is correct

Exercise 2. Let G be a group such that $x^2 = 1$, for each $x \in G$. Then:

(a) G is cyclic **(b)** G is abelian **(c)** $|G| = 2$ **(d)** $|x| = 2, \ \forall\, x \in G$ **(e)** $Z(G) = \{1\}$

Exercise 3. Let S be the group of permutations of a finite set and let α and β be any two distinct transpositions in S. Then:

(a) $|\alpha\beta| = 1$ **(b)** $|\alpha\beta| = 2$ **(c)** $|\alpha\beta| = 1$ or 2 **(d)** $|\alpha\beta| = 1$ or 3 **(e)** $|\alpha\beta| = 2$ or 3

Exercise 4. Let S_{10} denote the symmetric group of degree 10 and let $\sigma \in S_{10}$ with $\sigma = \alpha_1\alpha_2\cdots\alpha_k$ where the α_i's are disjoint r_i-cycles such that $r_i \geq 2$, for each i, and $r_1 + r_2 + \cdots + r_k = 10$. Let N and n denote the largest and smallest possible orders for σ, respectively. Then $N - n =$

(a) 10

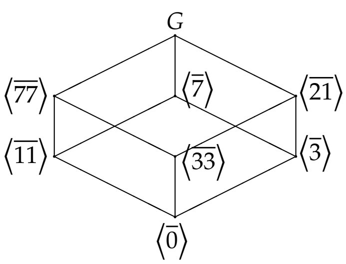
(b) 17

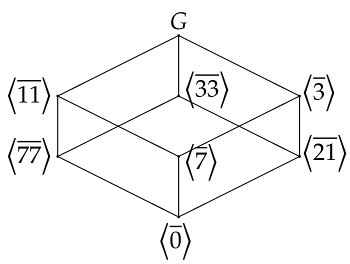
(c) 19

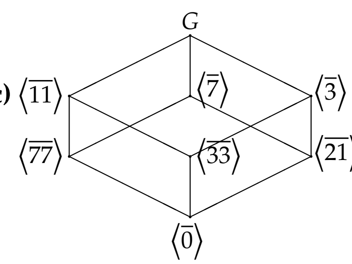
(d) 26

(e) 28

Exercise 5. The *lattice* of all subgroups of $\dfrac{\mathbb{Z}}{231\mathbb{Z}}$ is.

(a)


(b)


(c)


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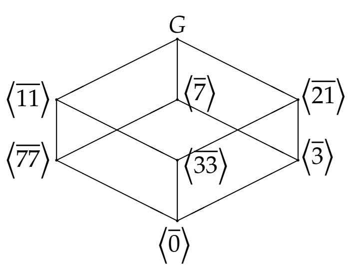
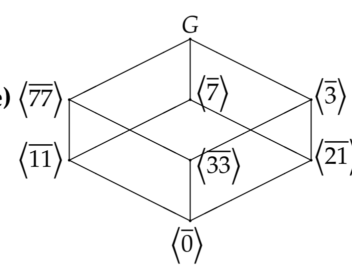


(e)


Exercise 6. $G := \{1, 4, 11, 14, 16, 19, 26, 29, 31, 34, 41, 44\}$ is a group under multiplication modulo 45. Then:

(a) G is the internal direct product of $\langle 4 \rangle$ and $\langle 19 \rangle$

(b) G is the internal direct product of $\langle 4 \rangle$ and $\langle 31 \rangle$

(c) G is the internal direct product of $\langle 11 \rangle$ and $\langle 19 \rangle$

(d) G is the internal direct product of $\langle 11 \rangle$ and $\langle 26 \rangle$

(e) G is the internal direct product of $\langle 16 \rangle$ and $\langle 26 \rangle$

Exercise 7. In the group $\frac{\mathbb{Z}}{25\mathbb{Z}} \oplus \frac{\mathbb{Z}}{5\mathbb{Z}} \oplus \frac{\mathbb{Z}}{5\mathbb{Z}}$, the number of elements of order 25 is equal to

(a) 0 **(b)** 100 **(c)** 125 **(d)** 480 **(e)** 500

Exercise 8. Let G be an abelian group of order 24, which has fourteen elements of order 6. Then, G is isomorphic to:

(a) $\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}$ **(b)** $\frac{\mathbb{Z}}{6\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}}$ **(c)** $\frac{\mathbb{Z}}{12\mathbb{Z}} \oplus \frac{\mathbb{Z}}{12\mathbb{Z}}$

(d) $\frac{\mathbb{Z}}{8\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}$ **(e)** $\frac{\mathbb{Z}}{12\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}$

Exercise 9. Let G be a group of order pq, where p and q are distinct primes. Suppose G has two nontrivial proper normal subgroups H and K such that $|H| \neq |K|$. Then, the number of elements of order pq in G is equal to:

(a) 0 **(b)** $pq-1$ **(c)** $pq-p-q$ **(d)** $pq-p-q+1$ **(e)** $(p-1)+(q-1)$

Exercise 10. The **smallest** (nonzero) positive integer k such that $\bar{k}$ is nilpotent in $\dfrac{\mathbb{Z}}{1440\mathbb{Z}}$ is equal to

(a) 28 **(b)** 30 **(c)** 32 **(d)** 36 **(e)** 45

Exercise 11. Let x and y denote, respectively, the **greatest** and the **smallest** 10-digit positive integers that are divisible by 8 but whose **last three digits are all distinct**. Then $x-y$ is equal to:

(a) 8,999,999,864

(b) 8,999,999,968

(c) 8,999,999,872

(d) 8,999,999,984

(e) 8,999,999,888

Exercise 12. Which one of the following statements is **WRONG**?

(a) $\dfrac{\mathbb{Z}}{2\mathbb{Z}}\big[X\big]$ is an infinite ring

(b) $\dfrac{\mathbb{Z}}{11\mathbb{Z}}\big[i\big]$ is a field

(c) $\dfrac{\mathbb{Z}}{5\mathbb{Z}}\big[i\big]$ is a finite ring

(d) $\dfrac{\mathbb{Z}}{13\mathbb{Z}}\big[i\big]$ is Not an integral domain

(e) $\dfrac{\mathbb{Z}}{7\mathbb{Z}}\big[i\big]$ is Not a field

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Exercise 13. Consider the commutative ring $R := \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$ and let

$$K := \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} \mid a \in \mathbb{Z} \right\} \text{ and } L := \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z} \right\}.$$

Then:

(a) K is a prime ideal of R

(b) $\dfrac{R}{K} \cong \dfrac{\mathbb{Z}}{2\mathbb{Z}}$

(c) L is a prime ideal of R and $\dfrac{R}{K} \cong \mathbb{Z}$

(d) $\dfrac{R}{K} \cong \mathbb{Z} \times \mathbb{Z}$

(e) L is a maximal ideal of R

Exercise 14. Which one of the following statements is WRONG?

(a) In $\mathbb{Z}[i]$, the principal ideal $(1-i)$ is maximal

(b) In $\dfrac{\mathbb{Z}}{5\mathbb{Z}}\left[i\right]$, the principal ideal $(1+i)$ is not prime

(c) $\dfrac{\mathbb{Z}[i]}{i\mathbb{Z}[i]} \cong \left\{0\right\}$

(d) In $\mathbb{Z}\times\dfrac{\mathbb{Z}}{6\mathbb{Z}}$, the ideal $2\mathbb{Z}\times\dfrac{\mathbb{Z}}{6\mathbb{Z}}$ is not maximal

(e) In $\dfrac{\mathbb{Z}}{2\mathbb{Z}}\times\dfrac{\mathbb{Z}}{2\mathbb{Z}}$, the ideal $0\times\dfrac{\mathbb{Z}}{2\mathbb{Z}}$ is principal

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Exercise 15. Consider the following polynomials:

$$f_1 = X^3+2X^2+X-1 \quad ; \quad f_2 = X^5-X^3+3X^2-3 \quad ; \quad f_3 = X^4+4X^2+6$$
$$f_4 = 2X^4+4X^2+2 \quad ; \quad f_5 = 2X^3+X^2+3X+2$$

Then, in $\mathbb{Q}[X]$

(a) f_1, f_2 are irreducible and f_4 is reducible

(b) f_1, f_3 are irreducible and f_5 is reducible

(c) f_1, f_5 are irreducible and f_2 is reducible

(d) f_1, f_5 are irreducible and f_3 is reducible

(e) f_3, f_5 are irreducible and f_1 is reducible

Exercise 16. The ring $\dfrac{\mathbb{Q}[X]}{(X^4-3X^2+2)}$ is isomorphic to:

(a) $\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}(i)$

(b) $\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}(\sqrt{2})$

(c) $\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}$

(d) $\mathbb{Q}\times\mathbb{Q}(i)\times\mathbb{Q}(\sqrt{2})$

(e) $\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}(\sqrt{2})\times\mathbb{Q}(\sqrt{2})$

Exercise 17. The polynomial $f := X^4 + 1$ is irreducible over

(a) $\dfrac{\mathbb{Z}}{2\mathbb{Z}}$ **(b)** $\mathbb{R}$ **(c)** $\mathbb{Q}$ **(d)** $\mathbb{Q}[i]$ **(e)** $\mathbb{Q}[\sqrt{2}]$

Exercise 18. Let $f = 1 + X + X^2 + \cdots + X^{p-1}$, where p is a prime number.
Then, in the ring $R := \dfrac{\mathbb{Q}[X]}{(f^p)}$

(a) $X^p - p$ and $X^p - 1$ are units

(b) $X^p - p$ and $X^p - 1$ are nilpotent

(c) $X^p - 1 = 0$ and $X^p - p$ is a unit

(d) $X^p - 1$ is nilpotent and $X^p - p$ is a unit

(e) $X^p - p$ is nilpotent and $X^p - 1$ is a unit

Exercise 19. Let $\left\{\overline{s_1}, \overline{s_2}, \ldots, \overline{s_n}\right\}$ denote the set of all distinct solutions of the equation $x^{66} = 1 \pmod{61}$, where $\mathbf{1} \leq s_1 < s_2 < \cdots < s_n \leq \mathbf{60}$. Then, $s_1 + s_2 + \cdots + s_n =$

(a) 61 **(b)** 122 **(c)** 183 **(d)** 244 **(e)** 366

Exercise 20. Let $G := \dfrac{\mathbb{Z}}{7\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{7\mathbb{Z}}$ and let n denote the number of subgroups of G of order 7. Then

(a) $n = 1$ **(b)** $n = 6$ **(c)** $n = 7$ **(d)** n = 8 **(e)** $n = 48$