King Fahd University of Petroleum & Minerals
Department of Mathematics

Semester (252) | Math 333 Final Exam | May 17, 2026 | 8:00-10:30 am

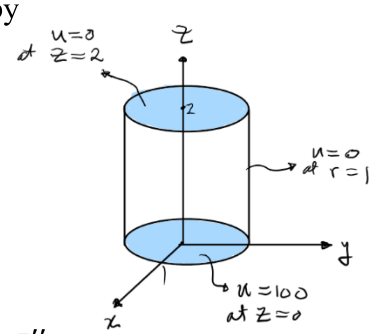
Name: ______________________________ I.D. # ____________________

<u>Points (Q1 = 6 , Q2 = 4 , Q3 = 5 , Q4 = 5 , Q5 = 5 , Q6 = 4 , Q7 = 6)</u>

1. Consider the vector field $\vec{F}(x,y,z) = 2z\,i - 3x\,j + 4y\,k$ and the upward oriented surface which is the portion of the paraboloid $z = 16 - x^2 - y^2$ for $z \geq 0$. **<u>Verify</u>** Stokes' theorem.

2. Find the half-range *<u>sine</u>* expansion of $f(x) = \cos x\,,\ 0 < x < \frac{\pi}{2}$. <u>Simplify your answer</u>.

3. The steady-state temperature in the circular cylinder shown is modeled by



$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0\,, \qquad 0 < r < 1, \qquad 0 < z < 2$$

$$u(1,z) = 0, \qquad u(r,0) = 100, \qquad u(r,2) = 0$$

Solve for $u(r,z)$. (With $u(r,z) = Z(z)R(r)$, consider only eigenvalues: $\frac{Z''}{Z} = +\lambda^2, \lambda > 0$)

4. a) Use Laplace transform to solve the boundary value problem

$$\frac{\partial u}{\partial x} + 2x\frac{\partial u}{\partial t} = 2x, \quad x > 0\,,\ t > 0$$
$$u(x,0) = 1, \qquad u(0,t) = 1$$

b) Sketch the solution $u(x,1)$ on the interval $x \epsilon [0,2]$.

5. a) Find the Fourier integral representation of $f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0 \end{cases}$. [Hint $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$ and $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$]

b) Based on your answer in (a), evaluate the integral $\int_0^\infty \frac{dw}{1+w^2}$.

6. The temperature, $u(x,t)$, in a semi-infinite rod is governed by the following BVP. Use an appropriate Fourier Transform to obtain $u(x,t)$. <u>Leaving the solution as an integral is ok</u>.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}\,, \qquad 0 < x < \infty, \qquad t > 0$$
$$u(x,0) = 0, \qquad u_x(0,t) = -1$$

7. It can be shown that $v(x,y) = \frac{\sin(\pi\,y)\sinh(\pi\,x)}{\sinh\pi}$ is a solution to the problem

$$v_{xx} + v_{yy} = 0\,, \quad 0 < x < 1, \quad 0 < y < 1$$
$$v(0,y) = v(x,0) = v(x,1) = 0, \ \ v(1,y) = \sin\pi y$$

Use separation of variables to solve the following problem.

$$u_{xx} + u_{yy} = 0\,, \quad 0 < x < 1, \quad 0 < y < 1$$

$$u(0,y) = 0, \quad u(1,y) = \sin\pi y\,, \quad u(x,0) = 0, \quad u(x,1) = x$$

Formulas

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt = \int_a^b \|\mathbf{r}'(t)\|\, dt.$$

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C (\mathbf{F}\cdot\mathbf{T})\,dS = \iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\,dS,$$

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\,dS = \iiint_D \text{div } \mathbf{F}\,dV.$$

$$\int x\cos(\alpha x)dx = \frac{\cos(\alpha x)}{\alpha^2} + \frac{x\sin(\alpha x)}{\alpha} + c \;,\; \int x\sin(\alpha x)dx = \frac{\sin(\alpha x)}{\alpha^2} - \frac{x\cos(\alpha x)}{\alpha} + c,$$

$$\int \sin(\alpha x)\cos(\beta x)dx = \frac{\cos((\beta-\alpha)x)}{2(\beta-\alpha)} - \frac{\cos((\beta+\alpha)x)}{2(\beta+\alpha)} + c$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt, \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad \mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad \mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2},$$

$$\mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2}, \quad \mathcal{L}\{\cosh kt\} = \frac{s}{s^2-k^2}, \quad \mathcal{L}\{e^{a\,t}\} = \frac{1}{s-a}, \quad \mathcal{L}\{e^{a\,t}f(t)\} = F(s-a),$$

$$\mathcal{L}\{f(t-a)\,\mathcal{U}(t-a)\} = e^{-a\,s}F(s), \quad \mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s), \quad f*g = \int_0^t f(\tau)\,g(t-\tau)\,d\tau,$$

$$\mathcal{L}\{f*g\} = F(s)\,G(s), \quad \mathcal{L}\{\delta(t-t_0)\} = e^{-s\,t_0}, \quad \mathcal{L}\{f(t)\} = \frac{1}{1-e^{sT}}\int_0^T e^{-s\,t} f(t)\,dt,$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots f^{(n-1)}(0)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{p} + b_n\sin\frac{n\pi x}{p}\right],$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}dx\,,\quad b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}dx$$

The **Fourier–Bessel series** of a function f defined on the interval $(0, b)$ is given by

(i)
$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x)$$
$$c_i = \frac{2}{b^2 J_{n+1}^2(\alpha_i b)}\int_0^b x J_n(\alpha_i x) f(x)\,dx,$$
where the α_i are defined by $J_n(\alpha b) = 0$.

(ii)
$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x)$$
$$c_i = \frac{2\alpha_i^2}{(\alpha_i^2 b^2 - n^2 + h^2)J_n^2(\alpha_i b)}\int_0^b x J_n(\alpha_i x) f(x)\,dx,$$
where the α_i are defined by $hJ_n(\alpha b) + \alpha b J_n'(\alpha b) = 0$.

(iii)
$$f(x) = c_1 + \sum_{i=2}^{\infty} c_i J_0(\alpha_i x)$$
$$c_1 = \frac{2}{b^2}\int_0^b x f(x)\,dx, \quad c_i = \frac{2}{b^2 J_0^2(\alpha_i b)}\int_0^b x J_0(\alpha_i x) f(x)\,dx,$$
where the α_i are defined by $J_0'(\alpha b) = 0$.

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$
$$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x).$$

The **Fourier–Legendre series** of a function f defined on the interval $(-1, 1)$ is given by
$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x),$$
where
$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx.$$

$$f(x) = \frac{1}{\pi}\int_0^\infty [A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x]\,d\alpha,$$
$$A(\alpha) = \int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx$$
$$B(\alpha) = \int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx.$$

$$f(x) = \frac{2}{\pi}\int_0^\infty A(\alpha)\cos\alpha x\,d\alpha,$$
$$A(\alpha) = \int_0^\infty f(x)\cos\alpha x\,dx.$$
$$f(x) = \frac{2}{\pi}\int_0^\infty B(\alpha)\sin\alpha x\,d\alpha,$$
$$B(\alpha) = \int_0^\infty f(x)\sin\alpha x\,dx.$$

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx = F(\alpha)$$
$$\mathcal{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha x}\,d\alpha = f(x)$$
$$\mathcal{F}_s\{f(x)\} = \int_0^\infty f(x)\sin\alpha x\,dx = F(\alpha)$$
$$\mathcal{F}_s^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_0^\infty F(\alpha)\sin\alpha x\,d\alpha = f(x)$$
$$\mathcal{F}_c\{f(x)\} = \int_0^\infty f(x)\cos\alpha x\,dx = F(\alpha)$$
$$\mathcal{F}_c^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_0^\infty F(\alpha)\cos\alpha x\,d\alpha = f(x)$$
$$\mathcal{F}\{f^{(n)}(x)\} = (-i\alpha)^n\,\mathcal{F}\{f(x)\} = (-i\alpha)^n F(\alpha),$$
$$\mathcal{F}_s\{f''(x)\} = -\alpha^2 F(\alpha) + \alpha f(0).$$
$$\mathcal{F}_c\{f''(x)\} = -\alpha^2 F(\alpha) - f'(0).$$