King Fahd University of Petroleum and Minerals
Department of Mathematics

Math 371

Final Exam

252

May 19, 2026

Net Time Allowed: 120 Minutes

MASTER VERSION

1. Consider the linear system $A\,x = b$

$$\text{where} \quad A = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}, \qquad b = \begin{bmatrix} 7 \\ 4 \end{bmatrix},$$

Let v_1 and v_2 denote the first two **search direction** vectors generated by the **conjugate gradient method** with the intial guess $X^{(0)} = (0,0)^t$. Compute $v_1^t\,A\,v_2 =$

(a) 0.000

(correct)

(b) 1.237

(c) 7.000

(d) −0.463

(e) 4.000

2. Let $A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ Which of the following is **FALSE?**

(a) v_1 and v_2 are linearly dependent vectors.

(correct)

(b) v_1 and v_3 are A-orthogonal.

(c) A is a symmetric matrix.

(d) v_1 is an eigenvector of the matrix A.

(e) $\lambda = 4$ is an eigenvalue of the matrix A.

3. If the Linear **Finite Difference method** is used to approximate the solution of the boundary value problem

$$y'' - 4\cos(\pi x)\ y' = 3\ \sin(\frac{\pi x}{2})\ y + \sin(\frac{\pi x^2}{2}), \quad 0 \le x \le 3, \quad y(0) = 1, \quad y(3) = 3,$$

with $h = 1$, then $y(2) \approx$

(a) -21

(correct)

(b) -13

(c) -3

(d) -2

(e) -34

4. Which of the following matrices are orthogonal matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

(a) A and B

(correct)

(b) A only

(c) C only

(d) B and C

(e) A, B and C

5. **A natural cubic spline** S is defined by

$$S(x) = \begin{cases} s_1(x) = 2 + (x-1)^3, \quad 1 \le x \le 2, \\ s_2(x) = 3 + 3\ (x-2) + a\ (x-2)^2 + b\ (x-2)^3, \quad 2 \le x \le 3. \end{cases}$$

If S interpolates the data $(1,2), (2,3)$ and $(3,c)$, then the value of c is equal to

(a) 8

____(correct)

(b) 4

(c) 5

(d) 2

(e) 1

6. Use the **composite Simpson's rule** with $n = 4$ to approximate the integral $\int_0^8 \frac{x}{x^2+1} dx \approx$

(a) 1.8949

____(correct)

(b) 2.0872

(c) 2.0002

(d) 1.5119

(e) 2.7047

7. Let $X^{(0)} = (0,0,0)^t$. If the second iteration of the **Gauss-Siedel method** for the linear system

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \quad \text{is} \quad X^{(2)}. \quad \text{Then} \quad || X^{(2)} ||_\infty =$$

(a) 0.5625

____(correct)

(b) 0.1250

(c) 0.4061

(d) 0.4633

(e) 0.1990

8. Let $\mathbf{x}^{(0)} = (0,\ 0)^t$. If the **Steepest Descent method** with $\alpha = 0.1$ is used to solve the non-linear system of equations

$$3x_1 + x_2 = 5$$

$$x_1^2 + 2x_2^2 = 9,$$

then the first iteration $(x_1,\ x_2)^t =$

(a) $(3,\ 1)^t$

____(correct)

(b) $(1,\ 2)^t$

(c) $(-36,\ 72)^t$

(d) $(30,\ 10)^t$

(e) $(-1,\ -2)^t$

9. Which of the following method – application pairs is incorrectly matched?

   (a) Simpson's rule – numerical differentiation (approximating derivatives).

   ____(correct)

   (b) Steepest descent method – Optimization (finding minima)

   (c) Conjugate gradient method – Solving linear systems

   (d) Finite difference method – solving boundary value problems (BVP).

   (e) Runge–Kutta method – solving initial value problems (IVP).

10. Use the **Gram–Schmidt process** to determine a set of orthogonal vectors $\{v_1, v_2, v_3\}$ from the linearly independent vectors

$$X_1 = (1, 1, 1)^t, \qquad X_2 = (2, 0, 4)^t, \qquad X_3 = (0, 4, 2)^t.$$

   (a) $\{q_1 = (1, 1, 1)^t, q_2 = (0, -2, 2)^t, q_3 = (-2, 1, 1)^t\}$

   ____(correct)

   (b) $\{q_1 = (1, 1, 1)^t, q_2 = (0, 2, -2)^t, q_3 = (4, 1, 2)^t\}$

   (c) $\{q_1 = (1, 1, 1)^t, q_2 = (0, -2, 2)^t, q_3 = (2, -1, -1)^t\}$

   (d) $\{q_1 = (1, 1, 1)^t, q_2 = (-2, 2, 0)^t, q_3 = (-2, 3, 3)^t\}$

   (e) $\{q_1 = (1, 1, 1)^t, q_2 = (0, -2, 2)^t, q_3 = (-2, 1, -1)^t\}$

11. Which of the following statements is **FALSE** about the **Conjugate Gradient Method (CG)** ?

(a) CG applies to any $n \times n$ square matrix A, whether symmetric or not.

(correct)

(b) The **search direction vectors** $\{v_1, v_2, v_3, \ldots\}$ in CG are mutually A-orthogonal.

(c) The **residual vectors** $\{r_0, r_1, r_2, \ldots\}$ in the CG are mutually orthogonal.

(d) When A is **symmetric positive definite matrix (SPD)**, CG minimizes the quadratic function $f(x) = x^T Ax - 2\, b^T x$.

(e) CG is useful when employed as an **iterative approximation method** for solving large **sparse systems**.

12. Consider solving the linear system

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

using **Conjugate Gradient method** with initial guess $\mathbf{x}^{(\mathbf{0})} = (0, 0, 0)^t$. Then the infinity norm of the first iteration $||\ \mathbf{x}^{(\mathbf{1})}\ ||_\infty =$

(a) 0.8571

(correct)

(b) 3.0638

(c) 2.6154

(d) 1.4315

(e) 1.0244

13. Given the following data points: $(0, 2.0)$, $(1, 3.2)$, $(3, 7.6)$. Construct the **least squares** approximation of the form

$$P(x) = b\,e^{a\,x}$$

Then $P(2.05) =$

(a) 5.0151

(correct)

(b) 7.2476

(c) 4.0924

(d) 4.9787

(e) 3.8844

14. If $P_4(x) = a_4\,x^4 + a_3\,x^3 + a_2\,x^2 + a_1\,x + a_0$ is the **least squares polynomial of degree 4** for the data points: $(0, 0.21)$, $(1, 1.31)$, $(2, -0.1)$, $(4, -2.4)$, $(5, 1.1)$, $(6, 3.2)$. Then $P_4(3) =$
(HINT: You may use the output of the following **MATLAB** code)

```
clear
x = [0 1 2 4 5 6];
y = [0.21 1.31 -0.1 -2.4 1.1 3.2];
p = polyfit(x,y,4)
p =
   -0.0898    1.2303   -4.9565    5.3725    0.1187
```

(a) -2.4275

(correct)

(b) 113.6649

(c) -1.0839

(d) -0.9934

(e) -1.9934

15. Consider the linear system

$$A\,w \;=\; b$$

obtained from applying the **linear finite difference method** with $h = 1$ to the boundary value problem

$$y'' = 10y' + 11y + 2x, \quad 0 \le x \le 4, \quad y(0) = 4,\ y(4) = -2,$$

Compute $\quad ||\,A\,||_\infty \;+\; ||\,b\,||_\infty \;=$

(a) 45

(correct)

(b) 43

(c) 54

(d) 22

(e) 36

16. Suppose the singular value decomposition of a matrix A is given by

$$USV^t = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & \sqrt{2}/4 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^t,$$

and let $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. If $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is the solution of the linear system $Ax = b$, then $x_1 + x_2 + x_3 =$

(a) 3

(correct)

(b) 4

(c) $4\sqrt{2}$

(d) $2\sqrt{2}$

(e) $-\sqrt{2}$

17. Let $\mathbf{x}^{(0)} = (2,2)^t$. If the first iteration of **Newton's method** for the nonlinear system

$$x_1^2 + x_2 = 5,$$
$$(x_1-1)^2 + (x_2-2)^2 = 2,$$

is $\mathbf{x}^{(1)}$, then $||\, \mathbf{x}^{(1)} \,||_\infty =$

(a) 2.5

____(correct)

(b) 3.5

(c) 1.5

(d) 5.5

(e) 3.6

18. Let $\mathbf{x}^{(0)} = (1,0)^t$ and $\alpha = 0.4$. If the second iteration of the **Steepest Descent method** for the function

$$g(x_1,x_2) = (x_1^2-1)^2 + (x_2-2)^2 + 10$$

is $\mathbf{x}^{(2)}$, then $||\, \mathbf{x}^{(2)} \,||_\infty =$

(a) 1.92

____(correct)

(b) 1.60

(c) 2.55

(d) 6.40

(e) 1.00

19. Let A be a 4×3 matrix with singular value decomposition

$$A = \begin{bmatrix} 3.664 & -4.752 & 0 \\ 0 & 0 & -0.120 \\ -4.752 & 6.436 & 0 \\ 0 & 0 & 0.160 \end{bmatrix} = \begin{bmatrix} 0.6 & 0 & 0.8 & 0 \\ 0 & -0.6 & 0 & 0.8 \\ -0.8 & 0 & 0.6 & 0 \\ 0 & 0.8 & 0 & 0.6 \end{bmatrix} \begin{bmatrix} 10 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.6 & 0 & 0.8 \\ -0.8 & 0 & 0.6 \\ 0 & 1 & 0 \end{bmatrix}^{\mathbf{t}}.$$

The rank-1 approximation of A is

(Hint: A_1 contains the most significant singular value)

(a) $A_1 = \begin{bmatrix} 3.6 & -4.8 & 0 \\ 0 & 0 & 0 \\ -4.8 & 6.4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(correct)

(b) $A_1 = \begin{bmatrix} 3.664 & -4.752 & 0 \\ 0 & 0 & 0 \\ -4.752 & 6.436 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $A_1 = \begin{bmatrix} 3.6 & -4.7 & 0 \\ 0 & 0 & 0 \\ -4.7 & 6.4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $A_1 = \begin{bmatrix} 3.6 & -4.8 & 0 \\ 0 & 0 & 0 \\ -4.8 & 6.2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(e) $A_1 = \begin{bmatrix} 3.6 & -4.8 & 0 \\ 0 & 0 & -0.1 \\ -4.8 & 6.4 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$

20. Given the matrix $A = \begin{bmatrix} -2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, its singular value decomposition (SVD) is $A = USV^t$. To help determine the matrices U, S and V, the eigenvalue of A^tA and AA^t are provided (via MATLAB outputs).

```
clear
A = [-2 0 ;1 0 ;0 1 ];
[V1,D1]= eig(A'*A)

V1 =                                    D1 =
    0    1                                      1      0
    1    0                                      0      5

[V2,D2] = eig(A*A')

V2 =                                    D2 =
   -0.4472         0   -0.8944                    0      0      0
   -0.8944         0    0.4472                    0      1      0
         0         1         0                    0      0      5
```

Using this information, compute the value of

$$U_{21} + S_{11} + V_{22} =$$

(a) 3.6833

______________________________ (correct)

(b) 1.2280

(c) 3.0361

(d) 4.3367

(e) 1.0255