King Fahd University of Petroleum & Minerals
Department of Mathematics
Math 513 Final Exam 252
Time Allowed: 180 Minutes

Name: ______________________________ ID#: ________________________

- The use of mobile phones and calculators is strictly prohibited during the exam.
- Show all steps clearly and indicate the correct final answer.

Question #	Marks	Maximum Marks
1		15
2		20
3		20
4		10
5		30
6		10
7		15
8		10
9		20
Total		150

Q:1 (15 points) Use the **eigenvalues and eigenvectors** approach to solve the following system of linear differential equations in the matrix form:

$$x' = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} x, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Note that: $x' = \frac{dx}{dt}$ and $x = x(t)$.

Q:2 ((10+5+5) points) Answer the following **three parts**:

A) **Find the Fourier series** for the function: $f(t) = t^2, \quad -\pi < t < \pi.$

B) **Use the result of Part A**, to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

C) **Use the result of Part A**, to derive the Fourier series for $f(t) = t, \ -\pi < t < \pi.$

Q:3 ((10+10) points) Answer the following **two parts**:

A) Use the definition of the Fourier transform to show that the Fourier transform of

$$f(t) = \begin{cases} 1, & |t| < a, \\ 0, & |t| > a, \end{cases} \qquad \textbf{is} \qquad F(\omega) = \frac{2\sin(\omega a)}{\omega}.$$

B) Use the **Parseval equality** and the Fourier transform for $f(t)$ shown in Part A to prove that

$$\int_{-\infty}^{\infty} \frac{\sin^2(ax)}{x^2} dx = \pi a.$$

Q:4 (10 points) By using the **Fourier transform**, find a particular solution of the differential equation

$$y'' + 3y' + 2y = e^{-t}H(t).$$

Note that: Using another method **is not accepted**.

Q:5 (10+10+10 points) Given the two functions:

$$f(t) = H(t+2) - H(t-2), \quad g(t) = e^{-t}H(t),$$

A) Find $f(t) * g(t)$.
B) Find $\mathcal{F}\{f(t)\}$, $\mathcal{F}\{g(t)\}$, and $\mathcal{F}\{f(t) * g(t)\}$.
C) Verify the Fourier convolution theorem of the above two functions $f(t)$ and $g(t)$.

Q:6 (10 points) Find the Laplace transform for the **periodic function**

$$f(t) = \begin{cases} 1, & 0 < t < a, \\ -1, & a < t < 2a, \end{cases} \qquad \text{with} \qquad f(t) = f(t+2a).$$

Q:7 (15 points) By using **Heaviside step functions**, find the Laplace transform of

$$f(t) = \begin{cases} t, & 0 \le t \le 2, \\ 4-t, & 2 \le t \le 4, \\ 0, & t \ge 4. \end{cases}$$

Note that: Using the definition of the Laplace Transform (integrating)**is not accepted**. So, you have to write f(t) in terms of the Heaviside functions, then use the suitable formula from the provided sheet.

Q:8 (10 points) Use the **Laplace shifting property**; $\mathcal{L}\{f(t-b)H(t-b)\}$ to find the Laplace transform of the function

$$f(t) = (t^2 - 1)H(t-1).$$

Note that: Using the definition of the Laplace Transform (integrating) **is not accepted**.

Q:9 (20 points) Use the **Laplace transform** to solve (find $y(t)$) the following initial value problem

$$y'' + 2ty' - 8y = 0$$

with the initial conditions

$$y(0) = 1, \qquad y'(0) = 0.$$

Note that: Using another method **is not accepted**.