

**King Fahd University of Petroleum & Minerals
(KFUPM)**

Department of Mathematics

MATH531 — Real Analysis

Final Exam

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Name: _____

ID: _____

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Question 1. (10 Points)

Let $1 < p, q, r < \infty$ satisfy

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1.$$

If $f \in L^p(E)$, $g \in L^q(E)$, and $h \in L^r(E)$, prove that

$$\int_E |fgh| \leq \|f\|_p \|g\|_q \|h\|_r.$$

Solution.

Indeed, put

$$s = \frac{qr}{q+r}.$$

Then

$$\frac{1}{s} = \frac{1}{q} + \frac{1}{r}.$$

By Hölder's inequality applied to g and h ,

$$\|gh\|_s \leq \|g\|_q \|h\|_r.$$

Also,

$$\frac{1}{p} + \frac{1}{s} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1.$$

Applying Hölder's inequality to f and gh gives

$$\int_E |fgh| \leq \|f\|_p \|gh\|_s \leq \|f\|_p \|g\|_q \|h\|_r.$$

This proves the result.

Question 2. (10 Points)

Let f be absolutely continuous and singular on $[a, b]$. Show that f is constant. Also show that the Lebesgue decomposition of a function of bounded variation is unique if the singular function is required to vanish at $x = a$.

Solution.

Since f is absolutely continuous on $[a, b]$, we have

$$f(x) - f(a) = \int_a^x f'(t) dt.$$

Since f is singular, $f' = 0$ a.e. on $[a, b]$. Hence

$$f(x) - f(a) = \int_a^x 0 dt = 0.$$

Therefore $f(x) = f(a)$ for all $x \in [a, b]$, so f is constant.

Now suppose a function F of bounded variation has two decompositions

$$F = g_1 + s_1 = g_2 + s_2,$$

where g_1, g_2 are absolutely continuous, s_1, s_2 are singular, and

$$s_1(a) = s_2(a) = 0.$$

Then

$$g_1 - g_2 = s_2 - s_1.$$

The left-hand side is absolutely continuous, while the right-hand side is singular. Hence $g_1 - g_2$ is both absolutely continuous and singular, so it is constant. At $x = a$,

$$g_1(a) - g_2(a) = F(a) - s_1(a) - F(a) + s_2(a) = 0.$$

Thus $g_1 - g_2 = 0$, so $g_1 = g_2$. Therefore also $s_1 = s_2$. Hence the decomposition is unique.

Question 3. (10 Points)

Let f be a step function on $[a, b]$. Find a formula for its total variation.

Solution.

Suppose f is constant on the intervals determined by

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Write

$$f(x) = c_i \quad \text{for } x \in (x_{i-1}, x_i), \quad i = 1, \dots, n.$$

At the points x_i , possible jumps may occur. The total variation of f is the sum of the sizes of all its jumps.

More precisely,

$$V_a^b(f) = |f(a^+) - f(a)| + \sum_{i=1}^{n-1} |f(x_i^+) - f(x_i^-)| + |f(b) - f(b^-)|.$$

If f is right-continuous on $[a, b)$ and has values c_i on $[x_{i-1}, x_i)$, then this becomes

$$V_a^b(f) = \sum_{i=1}^{n-1} |c_{i+1} - c_i| + |f(b) - c_n|.$$

In particular, if the endpoint values agree with the limiting values, then

$$V_a^b(f) = \sum_{i=1}^{n-1} |c_{i+1} - c_i|.$$

Thus the total variation of a step function is exactly the sum of the absolute values of its jumps.

Question 4. (12 Points)

Let $\{f_n\}$ be a sequence of integrable functions on \mathbb{R} . Show that $\{f_n\}$ is uniformly integrable and tight over \mathbb{R} if and only if for each $\varepsilon > 0$, there are positive numbers r and δ such that for each open subset \mathcal{O} of \mathbb{R} and each index n ,

$$m(\mathcal{O} \cap (-r, r)) < \delta \implies \int_{\mathcal{O}} |f_n| < \varepsilon.$$

Solution.

Assume first that $\{f_n\}$ is uniformly integrable and tight. Given $\varepsilon > 0$, choose $r > 0$ such that

$$\int_{\mathbb{R} \setminus (-r, r)} |f_n| < \frac{\varepsilon}{2} \quad \text{for all } n.$$

By uniform integrability, choose $\delta > 0$ such that whenever $m(A) < \delta$,

$$\int_A |f_n| < \frac{\varepsilon}{2} \quad \text{for all } n.$$

If \mathcal{O} is open and $m(\mathcal{O} \cap (-r, r)) < \delta$, then

$$\int_{\mathcal{O}} |f_n| = \int_{\mathcal{O} \cap (-r, r)} |f_n| + \int_{\mathcal{O} \setminus (-r, r)} |f_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Conversely, assume the stated condition holds. To prove uniform integrability, let $\varepsilon > 0$ and choose r, δ from the hypothesis. If A is measurable and $m(A) < \delta/2$, then by outer regularity there is an open set $\mathcal{O} \supset A$ such that

$$m(\mathcal{O}) < \delta.$$

Thus

$$m(\mathcal{O} \cap (-r, r)) < \delta,$$

and hence

$$\int_A |f_n| \leq \int_{\mathcal{O}} |f_n| < \varepsilon.$$

So $\{f_n\}$ is uniformly integrable.

To prove tightness, take

$$\mathcal{O} = (-\infty, -r) \cup (r, \infty).$$

Then

$$m(\mathcal{O} \cap (-r, r)) = 0 < \delta.$$

Therefore

$$\int_{\mathbb{R} \setminus [-r, r]} |f_n| = \int_{\mathcal{O}} |f_n| < \varepsilon.$$

Hence $\{f_n\}$ is tight.

Question 5. (16 Points)

Let $\{f_n\}$ and $\{g_n\}$ be sequences of measurable functions on a measurable set E with $m(E) < \infty$. Suppose

$$f_n \rightarrow f \quad \text{in measure on } E$$

and

$$g_n \rightarrow g \quad \text{in measure on } E.$$

1. Show that for any real numbers α, β ,

$$\alpha f_n + \beta g_n \rightarrow \alpha f + \beta g$$

in measure on E .

2. Suppose that g is finite a.e. on E . Show that

$$f_n g \rightarrow f g$$

in measure on E .

Solution.

1. Let $\varepsilon > 0$. We need to show that

$$m(\{x \in E : |\alpha f_n(x) + \beta g_n(x) - \alpha f(x) - \beta g(x)| > \varepsilon\}) \rightarrow 0.$$

If $\alpha = \beta = 0$, the result is trivial. Otherwise, using the triangle inequality,

$$|\alpha(f_n - f) + \beta(g_n - g)| \leq |\alpha||f_n - f| + |\beta||g_n - g|.$$

Hence

$$\{|\alpha(f_n - f) + \beta(g_n - g)| > \varepsilon\} \subset \left\{ |f_n - f| > \frac{\varepsilon}{2|\alpha|} \right\} \cup \left\{ |g_n - g| > \frac{\varepsilon}{2|\beta|} \right\},$$

where the terms with zero coefficients are omitted.

Therefore,

$$m(\{|\alpha(f_n - f) + \beta(g_n - g)| > \varepsilon\}) \leq m\left(\left\{ |f_n - f| > \frac{\varepsilon}{2|\alpha|} \right\}\right) + m\left(\left\{ |g_n - g| > \frac{\varepsilon}{2|\beta|} \right\}\right).$$

Both terms tend to 0 because $f_n \rightarrow f$ and $g_n \rightarrow g$ in measure. Hence

$$\alpha f_n + \beta g_n \rightarrow \alpha f + \beta g$$

in measure.

2. Since $f_n \rightarrow f$ in measure, we have $f_n - f \rightarrow 0$ in measure. We first show that

$$g(f_n - f) \rightarrow 0$$

in measure. Let $\varepsilon > 0$. Since g is finite a.e. and $m(E) < \infty$,

$$m(\{x \in E : |g(x)| > M\}) \rightarrow 0 \quad \text{as } M \rightarrow \infty.$$

For $M > 0$,

$$\{|g(f_n - f)| > \varepsilon\} \subset \{|g| > M\} \cup \{|f_n - f| > \varepsilon/M\}.$$

Hence

$$m(\{|g(f_n - f)| > \varepsilon\}) \leq m(\{|g| > M\}) + m(\{|f_n - f| > \varepsilon/M\}).$$

Taking $n \rightarrow \infty$, the second term tends to 0. Then letting $M \rightarrow \infty$, we get

$$m(\{|g(f_n - f)| > \varepsilon\}) \rightarrow 0.$$

Thus

$$f_n g - f g = g(f_n - f) \rightarrow 0$$

in measure, so $f_n g \rightarrow f g$ in measure.

Question 6. (17 Points)

Define the functions f and g on $[-1, 1]$ by

$$f(x) = x^{1/3},$$

and

$$g(x) = \begin{cases} x^2 \cos(\pi/2x), & x \neq 0, x \in [-1, 1], \\ 0, & x = 0. \end{cases}$$

1. Show that both f and g are absolutely continuous on $[-1, 1]$.

2. For the partition

$$P_n = \{-1, 0, 1/2n, 1/(2n-1), \dots, 1/3, 1/2, 1\}$$

of $[-1, 1]$, examine $V(f \circ g, P_n)$.

3. Show that $f \circ g$ fails to be absolutely continuous on $[-1, 1]$.

Solution.

(1) For $f(x) = x^{1/3}$, we have

$$f'(x) = \frac{1}{3}|x|^{-2/3}, \quad x \neq 0.$$

Since $|x|^{-2/3}$ is integrable on $[-1, 1]$, it follows that f is absolutely continuous.

For $x \neq 0$,

$$g'(x) = 2x \cos(\pi/2x) + \frac{\pi}{2} \sin(\pi/2x).$$

This derivative is bounded on $[-1, 1] \setminus \{0\}$, hence integrable. Also g is continuous at 0 because

$$|g(x)| \leq x^2 \rightarrow 0.$$

Thus g is absolutely continuous on $[-1, 1]$.

(2) For $m \geq 1$,

$$g\left(\frac{1}{2m}\right) = \frac{1}{(2m)^2} \cos(m\pi) = \frac{(-1)^m}{(2m)^2},$$

while

$$g\left(\frac{1}{2m-1}\right) = \frac{1}{(2m-1)^2} \cos\left(\frac{(2m-1)\pi}{2}\right) = 0.$$

Therefore

$$(f \circ g)\left(\frac{1}{2m}\right) = \frac{(-1)^m}{(2m)^{2/3}},$$

and

$$(f \circ g)\left(\frac{1}{2m-1}\right) = 0.$$

Hence the variation over P_n satisfies

$$V(f \circ g, P_n) \geq 2 \sum_{m=1}^n \frac{1}{(2m)^{2/3}}.$$

(3) Since

$$\sum_{m=1}^{\infty} \frac{1}{m^{2/3}} = \infty,$$

we get

$$V(f \circ g, P_n) \rightarrow \infty.$$

Thus $f \circ g$ is not of bounded variation on $[-1, 1]$. Since every absolutely continuous function is of bounded variation, $f \circ g$ is not absolutely continuous.

Question 7. (10 Points)

Let $p > 1$ and let $f > 0$ a.e. with $f, f^{-1} \in L^1([0, 1])$. Use Jensen's inequality to prove

$$\left(\int_{[0,1]} f \, dm \right) \left(\int_{[0,1]} \frac{1}{f} \, dm \right) \geq 1.$$

Determine the equality case.

Solution.

Apply Jensen's inequality to the convex function

$$\varphi(t) = -\log t, \quad t > 0.$$

Since φ is convex, we get

$$-\log \left(\int_{[0,1]} f \, dm \right) \leq \int_{[0,1]} -\log f \, dm.$$

Equivalently,

$$\log \left(\int_{[0,1]} f \, dm \right) \geq \int_{[0,1]} \log f \, dm.$$

Applying the same inequality to $1/f$ gives

$$\log \left(\int_{[0,1]} \frac{1}{f} \, dm \right) \geq \int_{[0,1]} \log \left(\frac{1}{f} \right) \, dm = - \int_{[0,1]} \log f \, dm.$$

Adding the two inequalities yields

$$\log \left(\int_{[0,1]} f \, dm \right) + \log \left(\int_{[0,1]} \frac{1}{f} \, dm \right) \geq 0.$$

Therefore

$$\log \left[\left(\int_{[0,1]} f \, dm \right) \left(\int_{[0,1]} \frac{1}{f} \, dm \right) \right] \geq 0,$$

and hence

$$\left(\int_{[0,1]} f \, dm \right) \left(\int_{[0,1]} \frac{1}{f} \, dm \right) \geq 1.$$

Since $-\log t$ is strictly convex on $(0, \infty)$, equality holds if and only if f is constant almost everywhere.

Question 8. (10 Points)

Let $1 \leq p < \infty$. Suppose that $\{f_n\}$ is a sequence in $L^p(E)$ such that

$$\sum_{n=1}^{\infty} \|f_{n+1} - f_n\|_p < \infty.$$

Show that there exists a function $f \in L^p(E)$ such that

$$f_n \rightarrow f \quad \text{in } L^p(E).$$

(Hint: show that $\{f_n\}$ is Cauchy in $L^p(E)$.)

Solution.

For $m > n$, by the triangle inequality,

$$\|f_m - f_n\|_p \leq \sum_{k=n}^{m-1} \|f_{k+1} - f_k\|_p.$$

Since the series $\sum_{k=1}^{\infty} \|f_{k+1} - f_k\|_p$ converges, the tail tends to 0. Hence $\{f_n\}$ is Cauchy in $L^p(E)$. Since $L^p(E)$ is complete, there exists $f \in L^p(E)$ such that

$$\|f_n - f\|_p \rightarrow 0.$$