King Fahd University of Petroleum & Minerals (KFUPM)

Department of Mathematics

MATH531 — Real Analysis

Midterm Exam

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Name: ______________________________

ID: ____________

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Question 1. (20 Points)

1. Show that a set E is measurable if and only if for each $\varepsilon > 0$, there exist a closed set F and an open set O such that

$$F \subseteq E \subseteq O \quad \text{and} \quad m^*(O \setminus F) < \varepsilon.$$

2. Show that any choice set for the rational equivalence relation ($x \sim y$ if $y - x \in \mathbf{Q}$) on a set of positive outer measure must be uncountably infinite.

Solution.

(1) ($\Rightarrow$) Assume E is measurable. Given $\varepsilon > 0$, by the outer and inner regularity of a measurable set, there exist an open set $O \supset E$ with $m^*(O \setminus E) < \varepsilon/2$ and a closed set $F \subset E$ with $m^*(E \setminus F) < \varepsilon/2$. Then $F \subset E \subset O$ and

$$O \setminus F = (O \setminus E) \cup (E \setminus F) \quad \Rightarrow \quad m^*(O \setminus F) \leq m^*(O \setminus E) + m^*(E \setminus F) < \varepsilon.$$

($\Leftarrow$) Conversely, assume that for each $\varepsilon > 0$ there exist $F \subset E \subset O$ with F closed, O open, and $m^*(O \setminus F) < \varepsilon$. Then $O \setminus E \subset O \setminus F$, hence $m^*(O \setminus E) < \varepsilon$. Hence E is measurable.

(2) Let A be a set with $m^*(A) > 0$ and let $S \subset A$ be a choice set (one representative from each equivalence class mod $\mathbb{Q}$). Then every $a \in A$ can be written as $a = s + q$ for some $s \in S$ and $q \in \mathbb{Q}$, so

$$A \subset \bigcup_{q \in \mathbb{Q}} (S + q).$$

If S were countable, then $S + q$ is countable for each q, and the union over countably many q would be countable, so A would be countable. But any countable set has outer measure 0. This contradicts $m^*(A) > 0$. Hence S must be uncountable.

Question 2. (15 Points)
Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E. Define E_0 to be the set of points $x \in E$ at which $\{f_n(x)\}$ converges. Is the set E_0 measurable?
Solution.

Yes. A point $x \in E$ belongs to E_0 if and only if the sequence is Cauchy at x: for every $k \in \mathbb{N}$ there exists N such that for all $m, n \geq N$, $|f_m(x) - f_n(x)| < 1/k$. Hence

$$E_0 = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{m,n \geq N} \left\{ x \in E : \ |f_m(x) - f_n(x)| < \frac{1}{k} \right\}.$$

Each set $\{x : \ |f_m - f_n| < 1/k\}$ is measurable because $f_m - f_n$ is measurable and the inverse image of an open set is measurable. Countable unions/intersections preserve measurability, so E_0 is measurable.

Question 3. (15 Points)

1. State Egoroff's Theorem.
2. Let E be a measurable set with $m(E) < \infty$ and let $f_n \to f$ a.e. on E. Prove that for every $\delta > 0$, we have

$$m\Big(\{x \in E : |f_n(x) - f(x)| > \delta\}\Big) \to 0 \quad \text{as } n \to \infty.$$

Solution.

(1) Egoroff's Theorem. If $m(E) < \infty$ and $f_n \to f$ a.e. on E, then for every $\varepsilon > 0$ there exists a measurable set $A \subset E$ with $m(A) < \varepsilon$ such that $f_n \to f$ uniformly on $E \setminus A$.

(2) Fix $\delta > 0$ and let $\varepsilon > 0$ be arbitrary. By Egoroff, choose $A \subset E$ with $m(A) < \varepsilon$ such that $f_n \to f$ uniformly on $E \setminus A$. Then there exists N such that for all $n \geq N$,

$$\sup_{x \in E \setminus A} |f_n(x) - f(x)| \leq \delta.$$

Therefore, for $n \geq N$,

$$\{x \in E : |f_n(x) - f(x)| > \delta\} \subset A,$$

so $m(\{|f_n - f| > \delta\}) \leq m(A) < \varepsilon$. Since ε is arbitrary, we conclude

$$m(\{|f_n - f| > \delta\}) \to 0.$$

Question 4. (15 Points)

Assume f is a measurable function on $\mathbb{R}$ such that f^2 is integrable, i.e.

$$\int_{\mathbb{R}} |f(x)|^2 \, dx < \infty.$$

Prove that

$$\sum_{k=1}^{\infty} m\Big(\{x \in \mathbb{R} : \; |f(x)| \geq k\}\Big) < \infty.$$

Solution.

For each $k \in \mathbb{N}$, apply Chebyshev's inequality to the nonnegative function $|f|^2$:

$$m(\{|f| \geq k\}) = m(\{|f|^2 \geq k^2\}) \leq \frac{1}{k^2} \int_{\mathbb{R}} |f(x)|^2 \, dx.$$

Summing over k gives

$$\sum_{k=1}^{\infty} m(\{|f| \geq k\}) \leq \left(\int_{\mathbb{R}} |f(x)|^2 \, dx \right) \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$ and $\int_{\mathbb{R}} |f|^2 < \infty$, the right-hand side is finite. Therefore,

$$\sum_{k=1}^{\infty} m(\{x \in \mathbb{R} : \; |f(x)| \geq k\}) < \infty.$$

Question 5. (15 Points)
Let $\{f_n\}$ be a sequence of integrable functions on E for which $f_n \to f$ a.e. on E and f is integrable over E. Show that

$$\int_E |f - f_n| \to 0 \quad \text{if and only if} \quad \lim_{n\to\infty} \int_E |f_n| = \int_E |f|.$$

Solution.

($\Rightarrow$) If $\int_E |f - f_n| \to 0$, then

$$\left| \int_E |f_n| - \int_E |f| \right| \le \int_E ||f_n| - |f|| \le \int_E |f_n - f| \to 0,$$

so $\int_E |f_n| \to \int_E |f|$.

($\Leftarrow$) Assume $\int_E |f_n| \to \int_E |f|$. Define

$$h_n := |f_n| + |f| - |f_n - f|.$$

By the triangle inequality, $|f_n - f| \le |f_n| + |f|$, hence $h_n \ge 0$. Also, since $f_n \to f$ a.e., we have $|f_n| \to |f|$ a.e. and $|f_n - f| \to 0$ a.e., so $h_n \to 2|f|$ a.e. By Fatou's Lemma,

$$2\int_E |f| = \int_E \liminf_{n\to\infty} h_n \le \liminf_{n\to\infty} \int_E h_n = \liminf_{n\to\infty} \left(\int_E |f_n| + \int_E |f| - \int_E |f_n - f| \right).$$

Using $\int_E |f_n| \to \int_E |f|$ gives

$$2\int_E |f| \le 2\int_E |f| - \limsup_{n\to\infty} \int_E |f_n - f|.$$

Therefore $\limsup_{n\to\infty} \int_E |f_n - f| \le 0$. Since the integrals are nonnegative, this forces $\int_E |f_n - f| \to 0$.

Question 6. (15 Points)
Let f be an integrable function on E. For any integer $n \geq 1$ define $E_n = \{x \in E : \ |x| \geq n\}$. Show that

$$\lim_{n\to\infty} \int_{E_n} f = 0.$$

Solution.

Method 1: The general continuity statement: if $E_n \downarrow E_\infty$ and f is integrable, then

$$\int_{E_n} f \to \int_{E_\infty} f,$$

Since $E_{n+1} \subset E_n$ for all n, the sequence $\{E_n\}$ is decreasing and

$$\bigcap_{n=1}^{\infty} E_n = \emptyset.$$

As f is integrable on E, then

$$\int_{E_n} f \to \int_{E_\infty} f = \int_{\emptyset} f = 0,$$

Method 2: Since f is integrable on E, we have $\int_E |f| < \infty$. Define $g_n := |f|\chi_{E_n}$. Then $0 \leq g_n \leq |f|$ and $g_n(x) \to 0$ pointwise because $E_n \downarrow \emptyset$. By the Dominated Convergence Theorem,

$$\int_E g_n = \int_{E_n} |f| \to 0.$$

Finally,

$$\left|\int_{E_n} f\right| \leq \int_{E_n} |f| \to 0,$$

so in particular $\int_{E_n} f \to 0$.