King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 535 Major Exam 1
Term (252)

Time Allowed: 120 Minutes

Name: ______________________________ ID#: ________________

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

Question #	Marks	Maximum Marks
1		20
2		20
3		20
4		20
5		20
Total		100

Q:1 (20 points)

(a) (5 pts) Show that **not every** metric on a vector space is induced by a norm. Support your answer with a suitable example.

(b) (15 pts) Show that the space of all continuous functions $C[0,1]$ equipped with the norm

$$\|f\| = \int_0^1 |f(t)|dt,$$

is **NOT** a Banach space.

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Q:2 (20 points) Let $1 \leq p < \infty$. The space ℓ^p is defined by

$$\ell^p := \left\{ x = (x_k)_{k=1}^{\infty} \subset \mathbb{R} \text{ or } \mathbb{C} \ : \ \sum_{k=1}^{\infty} |x_k|^p < \infty \right\},$$

equipped with the norm $\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}$. Show that $(\ell^p, \|\cdot\|_p)$ is a **Banach space and separable**.

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Q:3 (20 points)

1. **(Riesz's Lemma)** Let X be a normed linear space, and let Y and Z be subspaces of X such that Y is closed and Y is a proper subspace of Z. Prove that for every real number θ with $0 < \theta < 1$, there exists an element $z \in Z$ such that

$$\|z\| = 1 \quad \text{and} \quad \|z - y\| \geq \theta \quad \text{for all } y \in Y.$$

2. Use Riesz's Lemma to prove that if a normed linear space X has the property that the closed unit ball

$$M = \{x \in X : \|x\| \leq 1\}$$

is compact, then X is **finite-dimensional**.

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Q:4 (20 points)

a. Let T be the differential operator on the normed space X of all polynomials on $[0,1]$ with values in $(\mathbb{R}, |\cdot|)$ under the max norm. Show that T is an **unbounded** linear operator.

b. Let X and Y be normed spaces, and let $T : X \to Y$ be a linear operator. Prove that T is **continuous** if and only if T is **bounded**.

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Q:5 (20 points)

a. Let (X, d) be a metric space and let (x_n) and (y_n) be Cauchy sequences in X. Define a sequence (a_n) by

$$a_n = d(x_n, y_n).$$

Show that the sequence (a_n) **converges**.

b. Let X be a nonempty set and let $d : X \times X \to \mathbb{R}$ satisfy the following properties:

(M2) $d(x, y) = 0$ if and only if $x = y$.

(M3) $d(x, y) = d(y, x)$

(M4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Show that $d(x, y) \geq 0$ for all $x, y \in X$, i.e. **the nonnegativity of a metric** follows from (M2)–(M4).

c. Show that a Cauchy sequence in a normed space is bounded.

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