King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 535 Major Exam II
Term (252)

Time Allowed: 120 Minutes

Name: ________________________________ ID#: ____________________

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

Question #	Marks	Maximum Marks
1		24
2		24
3		24
4		24
5		24
Total		120

Q:1 (24 points)

(a) (14 pts) Let $\{T_n\}$ be a sequence of bounded linear operators

$$T_n : X \to Y,$$

where X is a Banach space and Y is a normed space. Suppose that for every $x \in X$, the set

$$\{T_n(x) : n \in \mathbb{N}\}$$

is bounded in Y. Prove that the sequence of operator norms $\{\|T_n\|\}$ is bounded; that is, show that there exists a constant $C > 0$ such that

$$\|T_n\| \leq C \qquad \text{for all } n \in \mathbb{N}.$$

(b) (10 pts) Show that the *Uniform Boundedness Principle* need not hold if X is only a normed space and is not complete.

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Q:2 (24 points)

(a) (12 pts) Let T be a linear operator on a normed space X.

(I) Suppose that T is bounded. What additional conditions are required to ensure that T is a closed operator? Prove that under these conditions, T is indeed closed.

(II) Show that boundedness of T alone does not necessarily imply that T is a closed operator. Provide a counterexample.

(b) (12 pts) Let X and Y be Banach spaces and let

$$T : D(T) \subset X \to Y$$

be a closed linear operator, where $D(T)$ denotes the domain of T. Suppose that $D(T)$ is closed in X. Prove that the operator T is bounded.

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Q:3 (24 points)

a. (12 points) Prove the following Hahn–Banach extension theorem:

Let X be a normed vector space, and let $M \subseteq X$ be a linear subspace. Let

$$f : M \to \mathbb{F}$$

be a bounded linear functional, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Prove that there exists a bounded linear functional

$$F : X \to \mathbb{F}$$

such that

$$F(x) = f(x) \quad \text{for all } x \in M,$$

and

$$\|F\| = \|f\|.$$

b. (12 points) Let X be a normed space and let $M \subset X$ be a closed subspace. Suppose that $\omega \in X \setminus M$. Prove that there exists a continuous linear functional

$$F \in X^*$$

such that

$$F(m) = 0 \quad \text{for all } m \in M,$$

and

$$F(\omega) = 1.$$

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Q:4 (24 points)

a. (10 pts) Show that the dual space of ℓ^1 is ℓ^∞.

b. (14 pts) Prove that if the dual space U^* of a normed space U is separable, then U itself is separable.

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Q:5 (24 points)

a. (12 pts) Show that if a normed space X has a linearly independent subset of n elements, then so does the dual space X^*.

b. (12 pts) Let X be a normed space. For every fixed $x \in X$, define a functional

$$g_x : X^* \to \mathbb{K}$$

by

$$g_x(f) = f(x), \qquad f \in X^*.$$

Show that g_x is a bounded linear functional on X^* (that is, $g_x \in X^{**}$) and that

$$\|g_x\| = \|x\|.$$

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