King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 535 Final Exam
Term (252)

<u>Time Allowed</u>: 180 Minutes

Name: ____ ID#: ____________________

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

Question #	Marks	Maximum Marks
1		30
2		30
3		30
4		30
Total		120

Q:1 (30 points)

(a) (10 pts) Let (x_n) be a sequence in a normed space X. Prove that strong convergence implies weak convergence and that both limits coincide. Show also that the converse is not true in general.

(b) (10 pts) Let X be a normed space and X^* its dual. Prove that if a sequence $(f_n) \subset X^*$ converges weakly, then it also converges in the weak* topology, and show that the converse is not true in general by providing a counterexample.

(c) (10 pts) Let (x_n) be a sequence in $C[a,b]$ such that $x_n \xrightarrow{w} x$ in $C[a,b]$. Prove that (x_n) converges pointwise to x on $[a,b]$, that is, show that $x_n(t) \to x(t)$ for every $t \in [a,b]$.

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Q:2 (30 points)

(a) (15 pts) Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction mapping, i.e., there exists a constant $\alpha \in (0, 1)$ such that

$$d(T(x), T(y)) \leq \alpha\, d(x, y) \quad \text{for all } x, y \in X.$$

Prove that T has a unique fixed point $\bar{x} \in X$.

(b) (15 pts) Let $K(x, y)$ be a measurable function defined on the square $S = \{(x, y) : a \leq x \leq b,\ a \leq y \leq b\}$, and suppose that $\int_a^b \int_a^b |K(x, y)|^2\, dx\, dy < \infty$. Let $g \in L^2(a, b)$. Consider the integral equation

$$f(x) = g(x) + \mu \int_a^b K(x, y) f(y)\, dy.$$

Using the Banach Contraction Mapping Theorem, prove that this equation has a unique solution $f \in L^2(a, b)$ for sufficiently small values of the parameter μ.

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Q:3 (30 points)

a. (5 pts) Show that the space $C[a, b]$ of continuous real-valued functions on the interval $[a, b]$, equipped with the supremum norm,

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|,$$

is not an inner product space.

b. (10 pts) Let $\mathcal{H}$ and $\mathcal{G}$ be Hilbert spaces, and let $T : \mathcal{H} \to \mathcal{G}$ be a bounded linear operator. Show that there exists a unique bounded linear operator $T^* : \mathcal{G} \to \mathcal{H}$, called the adjoint of T, such that

$$\langle Tx, y\rangle_{\mathcal{G}} = \langle x, T^* y\rangle_{\mathcal{H}} \quad \text{for all } x \in \mathcal{H},\ y \in \mathcal{G}.$$

c. (15 pts) Show that an operator T on a Hilbert space H is unitary if and only if it is an isometric isomorphism of H onto itself.

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Q:4 (30 points) Let T be a **bounded normal** operator on a Hilbert space $\mathcal{H}$, and let $\alpha > 0$.

(a) (10 pts) Show that $\ker(T) = \ker(T^*)$, and show that if $\|Tx\| \geq \alpha\|x\| \quad$ for all $x \in \mathcal{H}$, then $\ker(T) = \{0\}$.

(b) (10 pts) Show that T is invertible if and only if $\ker(T) = \{0\}$ and the range of T is closed, and show that if T is invertible, then T^{-1} is bounded and normal.

(c) (10 pts) Show that T is invertible if and only if there exists a constant $c > 0$ such that

$$\|Tx\| \geq c\|x\| \quad \text{for all } x \in \mathcal{H}.$$

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