MATH 601: Introduction to Stochastic Differential Equations

Course level:	Introductory graduate course
Contact hours:	38 hours
Schedule:	Sunday and Tuesday, 16:00–17:15
Teaching period:	23^{rd} August–10^{th} December
Prerequisites:	Single-variable and multivariable calculus, and elementary ordinary differential equations
Instructor:	Dr. Majdouline Borji

1 Course description

This course provides an introduction to stochastic differential equations for students with a basic knowledge of probability. Its main objective is to develop the mathematical foundations of stochastic calculus while emphasizing the interpretation of stochastic differential equations as dynamical systems subject to random perturbations.

The course begins with a review of the probabilistic notions required for the study of stochastic processes. Then, it introduces Gaussian stochastic processes, Brownian motion, stochastic integration, Itô's formula, and the basic theory of stochastic differential equations.

2 Course content and distribution of hours

Each session lasts 1 hour and 15 minutes. The course consists of 30 sessions in total, divided into the following parts:

Part	**Number of sessions**
I. Introduction	1
II. Probability and random variables	4
III. Gaussian processes and Conditional expectation	4
IV. Brownian motion	8
V. The Itô integral and Itô's formula	8
VI. Stochastic differential equations	3
VII. Quizzes	2
Total	**30**

The distribution of hours in the following plan is approximate:

Chapter I: Introduction — 1 hour

1. Why should we learn probability?
2. From random variables to stochastic processes.
3. Examples from physics.
4. Examples from biology.
5. What probability allows us to predict.

Chapter II: Probability spaces and random variables — 5 hours

1. Probability spaces.
2. Random variables.
3. Expectation, variance, and covariance.
4. Moments and moment-generating functions.
5. Independent random variables.

Chapter III: Gaussian Processes and Conditional Expectation — 5 hours

1. Stochastic processes: basic definitions.
2. Gaussian random variables.
3. Gaussian random vectors.
4. Gaussian processes.
5. Conditional expectation for Gaussian vectors.
6. Gaussian white noise.
7. Pre-Brownian motion.

Chapter IV: Brownian Motion — 8 hours

1. **Kolmogorov's continuity theorem.**
2. **Construction of Brownian motion.**
3. **Wiener measure.**
4. **Basic properties of Brownian motion.**

   Stationary increments, scaling, time-homogeneity, and qualitative properties of Brownian trajectories.

   Brownian motion as the scaling limit of symmetric random walks will be presented conceptually.

5. **Filtrations and adapted processes.**

   Information available up to time t, the natural filtration of Brownian motion, adapted stochastic processes, and independence of future Brownian increments from past information.

6. **Brownian motion as a martingale.**

   Discrete-time and continuous-time martingales, elementary examples, and the martingale property.

Chapter V: The Itô Integral and Itô's Formula — 10 hours

1. **Quadratic variation of Brownian motion.**

   Definition along partitions and interpretation of the heuristic multiplication rules.

2. **Elementary adapted processes.**

   Processes of the form
   $$H_t = \sum_{k=0}^{n-1} H_k \mathbf{1}_{(t_k, t_{k+1}]}(t),$$
   where H_k is $\mathcal{F}_{t_k}$-measurable.

3. **The Itô integral.**

   Motivation for stochastic integration and explanation of why the classical Riemann–Stieltjes integral is not sufficient for Brownian motion. Definition for elementary adapted processes. Linearity and expectation of the Itô integral.

4. **The Itô isometry.**

5. **Extension to square-integrable processes.**

6. **The martingale property.**

7. **Itô's formula.**

   Statement, interpretation, and applications of the one-dimensional Itô formula. Product rule, stochastic integration by parts, and Itô's formula for time-independent functions. Itô's formula in several dimensions may be introduced if time permits.

Chapter VI: Stochastic Differential Equations and Applications — 4 hours

1. **Definition and interpretation of an SDE.**

   Differential and integral formulations:
   $$\mathrm{d}X_t = b(t, X_t)\ \mathrm{d}t + \sigma(t, X_t)\ \mathrm{d}B_t.$$
   Interpretation of b as the drift coefficient and σ as the diffusion coefficient.

   Strong solutions and pathwise uniqueness. Statement of the basic existence and uniqueness theorem under Lipschitz and linear-growth assumptions, without proof.

2. **Explicitly solvable SDEs.**

   Brownian motion with drift and additive-noise equations:
   $$\mathrm{d}X_t = \mu\ \mathrm{d}t + \sigma\ \mathrm{d}B_t.$$
   The Ornstein–Uhlenbeck process:
   $$\mathrm{d}X_t = -\lambda X_t\ \mathrm{d}t + \sigma\ \mathrm{d}B_t,$$
   together with its explicit solution, expectation, variance, and mean-reverting behavior.

3. **Application to mathematical finance (if time permits)**

   Modelling an asset price by geometric Brownian motion:
   $$\mathrm{d}S_t = \mu S_t\ \mathrm{d}t + \sigma S_t\ \mathrm{d}B_t.$$
   Introduction to the Black–Scholes model, the delta-hedging principle, and derivation of the Black–Scholes equation. European call payoff and statement of the Black–Scholes pricing formula.

3 Teaching schedule

The course runs from August 19 to December 10, 2026. Classes are held on Sundays and Tuesdays from 4:00 p.m. to 5:15 p.m. The course consists of 30 teaching sessions of 75 minutes each.

After each of the first four chapters, there will be a 30-minute quiz covering the topics studied in that chapter. The questions will be selected from the exercises provided at the end of the corresponding chapter. The approximate quiz dates are indicated in the teaching schedule below.

Session	**Date**	**Content**
1	Sunday, August 23	Course Syllabus and introduction: deterministic and stochastic models.
2	Tuesday, August 25	Probability spaces, random variables, and distributions.
3	Sunday, August 30	Expectation, variance, covariance, moments, and independence.
4	Tuesday, September 1	Moment-generating functions and modes of convergence.
5	Sunday, September 6	**Quiz 1**–Introduction to stochastic processes and sample paths.
6	Tuesday, September 8	Gaussian random variables and Gaussian random vectors.
7	Sunday, September 13	Gaussian processes and conditional expectation for Gaussian vectors.
8	Tuesday, September 15	Gaussian white noise and pre-Brownian motion.
9	Sunday, September 20	**Quiz 2**–Kolmogorov's continuity theorem and construction of Brownian motion.
10	Tuesday, September 22	Wiener measure and the definition of Brownian motion.
11	Sunday, September 27	Gaussian increments, independent increments, mean, and covariance of Brownian motion.
12	Tuesday, September 29	Scaling, stationary increments, and properties of Brownian paths.
13	Sunday, October 4	Filtrations, adapted processes, and the natural Brownian filtration.
14	Tuesday, October 6	Discrete-time and continuous-time martingales.
15	Sunday, October 11	Brownian motion as a martingale and elementary examples.
16	Tuesday, October 13	**Quiz 3**–Review and problem-solving session.
17	Sunday, October 18	Quadratic variation of Brownian motion and heuristic multiplication rules.
		Midterm break: October 20–22
18	Sunday, October 25	Elementary adapted processes and definition of the Itô integral.
19	Tuesday, October 27	Linearity, expectation, and basic properties of the Itô integral.
20	Sunday, November 1	The Itô isometry.
21	Tuesday, November 3	Extension of the Itô integral to square-integrable processes.
22	Sunday, November 8	Martingale property of the Itô integral and basic computations.
23	Tuesday, November 10	Itô's formula: statement and interpretation.

Session	Date	Content
24	Sunday, November 15	Applications of Itô's formula.
25	Tuesday, November 17	Product rule, stochastic integration by parts, and further examples.
		Autumn break: November 22–23
26	Tuesday, November 24	**Quiz 4**–Review and problem-solving session.
27	Sunday, November 29	Definition and interpretation of stochastic differential equations; statement of the existence and uniqueness theorem.
28	Tuesday, December 1	Explicitly solvable SDEs: additive noise and geometric Brownian motion.
29	Sunday, December 6	The Ornstein–Uhlenbeck process and mean-reverting behavior.
30	Tuesday, December 8	Application to mathematical finance: the Black–Scholes model and equation.

Remark 3.1. The schedule is intentionally flexible.

4 Assessment

The final grade is calculated out of 100 points. The assessment structure is as follows:

Component	Weight
4 Quizzes	40%
Final examination	60%
Total	**100%**

After each chapter, there will be a 30-minute quiz covering the topics studied in that chapter. The questions will be selected from the exercises provided at the end of the chapter. corresponding chapter:

- Quiz 1: Probability spaces and random variables;
- Quiz 2: Conditional Expectation and Gaussian Random vectors;
- Quiz 3: Brownian motion;
- Quiz 4: Itô integration and Itô's formula.

5 Main references

The course will be based primarily on the following references:

B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications*, 6th edition, Springer, Berlin–Heidelberg, 2003.

J.-F. Le Gall, *Brownian Motion, Martingales, and Stochastic Calculus*, Graduate Texts in Mathematics, Vol. 274, Springer, Cham, 2016.

These books will serve as general references rather than being followed literally. Since both texts are written at a relatively advanced and technical level, the material will be carefully selected and adapted to the level of the course. The presentation will emphasize intuition, concrete examples, and applications while avoiding the technical foundations of measure theory.

Complete lecture notes will be prepared and provided by the instructor, together with a selection of exercises specifically designed for this course. These lecture notes and exercises will contain all the material required to study for the quizzes and examinations. Students may consult the two references above for additional explanations and further reading, but they will not be required for assessment preparation.

6 Attendance policy

- Students are allowed a maximum of three absences during the semester. Starting from the fourth absence, each additional absence will result in a deduction of one point from the student's total course grade.

  Exceptions may be considered in documented cases of chronic illness or hospitalization for consecutive days.

- Arriving more than 15 minutes late will be counted as an absence.

7 Use of electronic devices

- The use of mobile phones is prohibited during class. Tablets are permitted only for taking notes and consulting course materials.
- After three warnings, any further violation of this policy will result in a two-point deduction from the classwork grade.