AS201- Major Exam 1

KFUPM, Department of Mathematics and Statistics

Kroumi Dhaker, Term 221

1 Exercise 1(15 points)

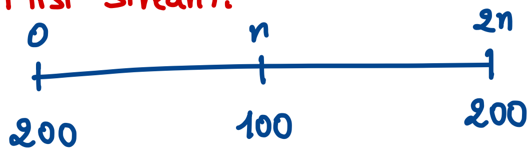
David can receive one of the following two payment streams:

- 200 at time 0, 100 at time n, and 200 at time $2n$.
- 500 at time 10.

At an effective annual interest rate of i, the present values of the two streams are equal. Given $v^n = 0.75941$, determine i

1. 1.24%
2. 2.76%
3. 2.12%
4. 2.48%
5. 3.24%

First stream:



$PV = 200 + 100v^n + 200v^{2n}$

$= 391.28\$$

Second stream

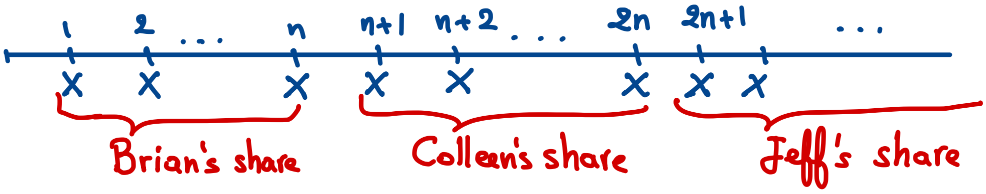
$PV = 500v^{10} = \frac{500}{(1+i)^{10}} \Rightarrow (1+i)^{10} = \frac{500}{PV}$

$\Rightarrow i = \sqrt[10]{\frac{500}{PV}} - 1 = 0.0248$

2 Exercise 2(15 points)

A perpetuity-immediate pays X per year. Brian receives the first n payments, Colleen receives the next n payments, and Jeff receives the remaining payments. Brian's share of the present value of the original perpetuity is 30%, and Colleen's share is K. Calculate K.

1. 21%
2. 49%
3. 29%
4. 31%
5. 35%



PV of the perpetuity $= \frac{X}{i}$

PV of Brian's share $= X a_{\overline{n}|i} = X \frac{1-v^n}{i} = 0.3 \times \frac{X}{i}$ $\Big\}$ $1-v^n = 0.3 \Rightarrow v^n = 0.7$

PV of Colleen's share $= X a_{\overline{n}|i} \times v^n = 0.3 \frac{X}{i} \times 0.7 = 0.21 \times \frac{X}{i} \Rightarrow K = 0.21$

3 Exercise 3(10 points)

An investment of 1000 accumulates to 1400 at the end of 5 years. If the force of interest is δ during the first year and 1.5δ in each subsequent year, find the equivalent effective annual interest rate in the first year.

1. 4.56%
2. 4.92%
3. 5.23%
4. 5.88%
5. 6.16%

$$A(5) = A(0)\, e^{\int_0^5 \delta_t\, dt},$$

where $\int_0^5 \delta_t\, dt = \int_0^1 \delta\, dt + \int_1^5 1.5\delta\, dt$

$$= \delta + 4 \times 1.5\delta = 7\delta$$

$$\Rightarrow A(5) = A(0) e^{7\delta} \Rightarrow e^{7\delta} = \frac{A(5)}{A(0)} = 1.4$$

$$\Rightarrow \delta = \frac{\ln(1.4)}{7} = \ln\left(1.4^{\frac{1}{7}}\right)$$

$$A(1) = A(0)\, e^{\int_0^1 \delta_t dt} = A(0)(1+i)$$

$$\Rightarrow i = e^{\int_0^1 \delta_t dt} - 1 = e^{\delta} - 1 = 1.4^{\frac{1}{7}} - 1 \simeq 0.0492$$

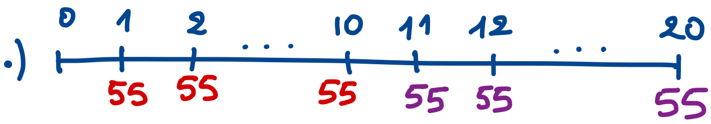
4 Exercise 4(10 points)

At an effective annual interest rate of i, $i > 0$, both of the following annuities have a present value of X:

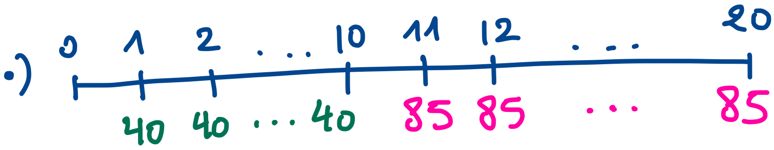
- a 20-year annuity-immediate with annual payments of 55.
- a 20-year annuity-immediate with annual payments that pays 40 per year for the first 10 years, and 85 per year for the final 10 years.

Calculate X.

1. 322.36
2. 412.85
3. 490
4. 510
5. 574.72



$$PV = 55\, a_{\overline{10}|i} + 55\, a_{\overline{10}|i} \cdot v^{10}$$



$$PV = 40\, a_{\overline{10}|i} + 85\, a_{\overline{10}|i} \cdot v^{10}$$

$$55\, a_{\overline{10}|i} + 55\, a_{\overline{10}|i} \cdot v^{10} = 40\, a_{\overline{10}|i} + 85\, a_{\overline{10}|i} \cdot v^{10}$$

$$\Rightarrow 55 + 55 v^{10} = 40 + 85 v^{10}$$

$$\Rightarrow v^{10} = 0.5 \Rightarrow \frac{1}{(1+i)^{10}} = 0.5 \Rightarrow i = \sqrt[10]{2} - 1$$

$$PV = 55\, a_{\overline{20}|i} = 55 \times \frac{1 - v^{20}}{i} = 55 \times \frac{1 - 0.5^{2}}{\sqrt[10]{2} - 1}$$

$$\simeq 574.72$$

5 Exercise 5(15 points)

At an effective annual interest of i, $i > 0$, the present value of a perpetuity paying 10 at the end of each 3-year period, with the first payment at the end of year 6, is 32. At the same effective annual rate of i, the present value of a perpetuity immediate paying 1 at the end of each 4-month period is X. Calculate X.

1. 16.28
2. 20.45
3. 27.27
4. 39.84
5. 43.12

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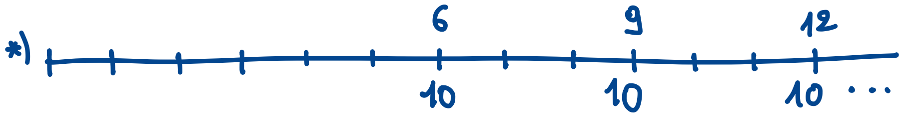

$32 = \frac{10}{j_1} v_{j_1}$ where j_1 is the effective interest rate per three years given by $1+j_1 = (1+i)^3 \Rightarrow 1+i = (1+j_1)^{1/3}$

$\Rightarrow 32 = \frac{10}{j_1(1+j_1)} \Rightarrow j_1^2 + j_1 - \frac{10}{32} = 0$

$\Delta = 1^2 + 4 \times \frac{10}{32} = \frac{72}{32} \Rightarrow \begin{cases} j_1 = \frac{-1-\sqrt{\frac{72}{32}}}{2} < 0 \text{ to reject} \\ j_1 = \frac{-1+\sqrt{\frac{72}{32}}}{2} > 0 \text{ to accept} \end{cases}$

$= 0.25$

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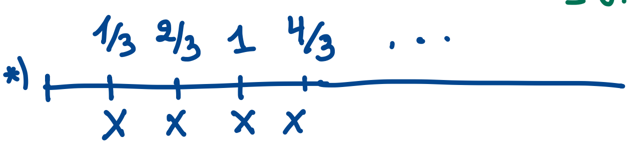
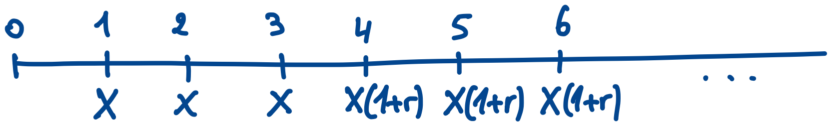


$X = \frac{1}{j_2}$ where j_2 is the effective interest rate per 4 months given by $j_2 = (1+i)^{\frac{1}{3}} - 1 = (1+j_1)^{\frac{1}{9}} - 1$

$\Rightarrow X = \frac{1}{j_2} = \frac{1}{(1+j_1)^{\frac{1}{9}} - 1} = \frac{1}{1.25^{1/9} - 1} = 39.83$

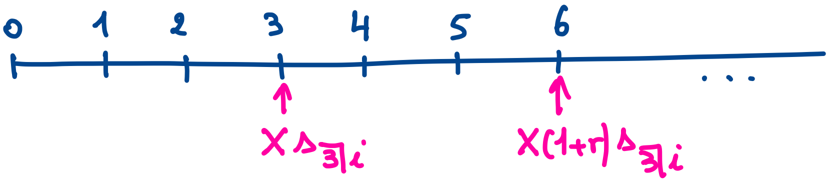
6 Exercise 6(20 points)

Smith has 100,000 with which she buys a perpetuity on January 1, 2005. Suppose that $i = 0.045$ and the perpetuity has annual payment beginning January 1, 2006. The first three payments are 2000 each, the next three payments are $2000(1+r)$ each, ..., increasing forever by a factor of $1+r$ every three years. What is r?



where $X = 2000$.

It can be written as



Then, $PV = \dfrac{X s_{\overline{3}|i}}{j - r}$ where $j = (1+i)^3 - 1 = 1.045^3 - 1$

$$\Rightarrow j - r = \frac{X s_{\overline{3}|i}}{PV} \Rightarrow r = j - \frac{X s_{\overline{3}|i}}{PV}$$

$$= 1.045^3 - 1 - \frac{2000 \times \dfrac{1.045^3 - 1}{0.045}}{100{,}000}$$

$$= 0.0784256$$

7 Exercise 7(15 points)

A customer if offered an investment where interest is calculated according to the following force of interest $\delta_t = 0.02t$ for $0 \leq t \leq 3$ and $\delta_t = 0.045$ for $t > 3$. The customer invests 1000 at time $t = 0$. What nominal rate of interest, compounded quarterly, is earned over the first four-year period?

*) $A(4) = A(0)\, e^{\int_0^4 \delta_t dt}$, where

$$\int_0^4 \delta_t dt = \int_0^3 0.02t\, dt + \int_3^4 0.045 dt$$

$$= \left[0.01t^2\right]_{t=0}^{t=3} + 0.045 = 0.135$$

$$\Longrightarrow A(4) = A(0) e^{0.135}$$

*) $A(4) = A(0)\left[1 + \frac{i^{(4)}}{4}\right]^{4\times 4} \Longrightarrow \left(1 + \frac{i^{(4)}}{4}\right)^{16} = e^{0.135}$

$$\Longrightarrow i^{(4)} = 4\left[\sqrt[16]{e^{0.135}} - 1\right] = 0.0339.$$

(3.39%)