# AS201- Final Exam (Code 1)

KFUPM, Department of Mathematics

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## Exercise 1

The current price of an annual coupon bond is 100. The yield to maturity is an annual effective rate of 8%. The derivative of the price of the bond with respect to the yield to the maturity −800.

Using the bond's yield rate, calculate the Macaulay duration of the bond is years.

1. 7.49
2. 7.56
3. 7.69
4. 8.00
5. 8.64

$P = 100$ ; $i = 8\%$ ; $\frac{dP}{di} = -800$

$$D = -(1+i)\frac{\frac{dP}{di}}{P} = 1.08 \times \frac{800}{100}$$

$$= 8.64$$

## Exercise 2

A common stock pays dividends at the end of each year into perpetuity. Assume that the dividend increases by 4% each year.

Using an annual effective interest rate of 8%, calculate the Macaulay duration of the stock in years

1. 25
2. 27
3. 35
4. 44
5. 52

$r = 4\%$ ; $i = 8\%$

Let $K$ be the initial payment

$K \quad K(1+r) \quad K(1+r)^2 \quad \ldots$ geometric perpetuity

$$P = \frac{K}{i-r} \Rightarrow \frac{dP}{di} = -\frac{K}{(i-r)^2}$$

$$\Rightarrow D = -(1+i)\frac{\frac{dP}{di}}{P} = (1+i)\frac{\frac{K}{(i-r)^2}}{\frac{K}{i-r}} = \frac{1+i}{i-r}$$

$$= 27$$

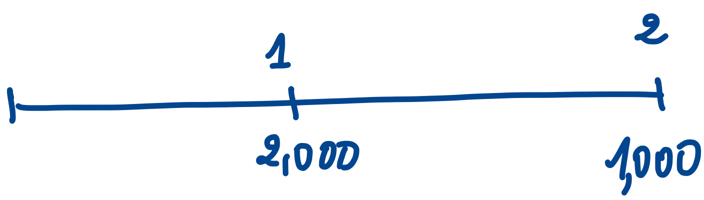
## Exercise 3

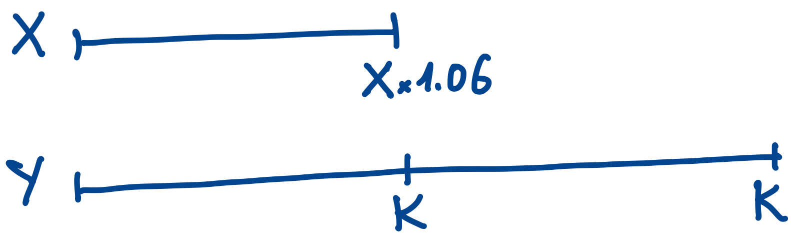
Joe must pay liabilities of 2,000 due one year from now and another 1,000 due two years from now. Here exactly matches his liabilities with the following two investments:

- Mortgage $I$: A one-year mortgage in which $X$ is lent. It is repaid with a single payment at time one. The annual effective interest rate is 6%.
- Mortgage $II$: A two-year mortgage in which $Y$ is lent. It is repaid with two equal annual payments. The annual effective interest rate is 7%.

Calculate $X+Y$.

1. 2,600
2. 2,682
3. 2,751
4. 2,825
5. 3,000




$$Y = K a_{\overline{2}|0.07}$$

$$\begin{cases} K = 1000 \\ K + X \cdot 1.06 = 2000 \end{cases} \Rightarrow \begin{cases} K = 1000 \\ X = 943.4 \end{cases}$$

$$\Rightarrow \begin{cases} Y = K a_{\overline{2}|0.07} = 1808 \\ X = 943.4 \end{cases} \Rightarrow X + Y = 2751.4 \$$$

## Exercise 4

A liability of a series of 15 annual payments of 35,000 with the first payment to be made one year from now.

The assets available to immunize this liability are five-year and ten year zero-coupon bonds.

The annual effective interest rate used to value the assets and the liability is 6.2%. The liability has the same present value and duration as the asset portfolio. Calculate the amount invested in the five-year zero-coupon bonds.

1. 127,974
2. 167,842
3. 208,566
4. 247,244
5. 292,361


$i = 6.2\%$

Let X and Y be the amounts invested in 5-year and 10-year zero-coup bonds, respectively. We have

$$X + Y = 35,000\, a_{\overline{15}|0.062} = 335,530.3 \quad (1)$$

$$5X + 10Y = 35,000 \times (Ia)_{\overline{15}|0.062}$$

$$= 35,000 \times \frac{\ddot{a}_{\overline{15}|0.062} - 15 \times 1.062^{-15}}{0.062}$$

$$= 2,312,522 \quad (2)$$

$10 \times (1) - (2)$ $\quad 5X = 1,042,781$

$$\Rightarrow X = 208,556.2$$

## Exercise 5

Kylie bought a 7-year, 5,000 par value bond with an annual coupon rate if 7.6% paid semiannually. She bought the bond with no premium or discount. $j = r = 3.8\%$

Calculate the Macaulay duration of this bond with respect to the yield rate on the bond.

1. 5.16
2. 5.35
3. 5.56
4. 5.77
5. 5.99

$$P = Fr\left[v + v^2 + \cdots + v^{14}\right] + Fv^{14}$$

$$D = \frac{Fr\left[1 \times v + 2 \times v^2 + \cdots + 14 \times v^{14}\right] + 14Fv^{14}}{Fr\left[v + v^2 + \cdots + v^{14}\right] + Fv^{14}}$$

$$= \frac{r(Ia)_{\overline{14}|r} + 14v^{14}}{r\, a_{\overline{14}|r} + v^{14}} = \frac{\ddot{a}_{\overline{14}|r} - 14v^{14} + 14v^{14}}{1 - v^{14} + v^{14}}$$

$$= \ddot{a}_{\overline{14}|r} = 11.11 \text{ (in terms of 6 months)}$$

$$\Rightarrow D = 5.56 \text{ (in terms of years)}$$

## Exercise 6

Krishna buys an $n$-year $1,000$ bond at par. The Macaulay duration is 7.959 years using an annual effective interest rate of 7.2%.

Calculate the estimated price of the bond, using the first-order modified approximation, if the interest rate rises to 8.0%

1. 940.60
2. 942.88
3. 944.56
4. 947.03
5. 948.47

$r = j = 7.2\%$ ; $P = 1000$; $D = 7.959$

$h = 8\% - 7.2\% = 0.8\%$

$$P(8\%) \approx P(7.2\%) \times [1 - h \times DM]$$

$$\approx P(7.2\%) \times \left[1 - h \times \frac{D}{1+j}\right]$$

$$\simeq 1000 \times \left[1 - 0.008 \times \frac{7.959}{1.072}\right]$$

$$\simeq 940.60$$

## Exercise 7

You are given the following information about an investment account

| Date | Account $K$ before activity | Deposit | Withdrawal |
|---|---|---|---|
| January 1, 1999 | 100 | | |
| July 1, 1999 | 125 | | X |
| October 1, 1999 | 110 | 2X | |
| December 31, 1999 | 125 | | |

| Date | Account $L$ before activity | Deposit | Withdrawal |
|---|---|---|---|
| January 1, 1999 | 100 | | |
| July 1, 1999 | 125 | | X |
| December 31, 1999 | 105.8 | | |

During 1999, the dollar weighted return for investment account $K$ equals the time weighted return for investment account $L$, which equals $i$.

Calculate $i$

1. 10.0%
2. 11.4%
3. 12.9%
4. 13.7%
5. 15.0%

$$i_K = \frac{125 - [100 + 2X - X]}{100 - X \times \frac{1}{2} + 2X \times \frac{1}{4}} = \frac{25 - X}{100}$$

$$i_L = \frac{125}{100} \times \frac{105.8}{125 - X} - 1 = \frac{13{,}225}{12{,}500 - 100X} - 1$$

$$i_K = i_L \Rightarrow \frac{25 - X}{100} = \frac{13{,}225}{12{,}500 - 100X} - 1$$

$$\Rightarrow (125 - X)(12{,}500 - 100X) = 100 \times 13{,}225$$

$$\Rightarrow 100X^2 - 25{,}000X + 240{,}000 = 0$$

$$\Rightarrow X = 10$$

$$\Rightarrow i = \frac{25 - X}{100} = 0.15$$

## Exercise 8

The table below defines available zero-coupon bonds and their prices

| Years to Maturity | Bond Price (Per Bond) | Redemption Value (Per Bond) |
|---|---|---|
| 1 | 961.54 | 1,000 |
| 2 | 966.14 | 1,000 |
| 3 | 878.41 | 1,000 |

A company chooses to purchase 15 one-year zero-coupon bonds, 20 two-year zero coupon bonds and 30 three-year zero-coupon bonds. Calculate the Macaulay duration of this portfolio.

1. 1.97
2. 2.11
3. 2.20
4. 2.33
5. 2.51

$D_1 = 1; \quad D_2 = 2; \quad D_3 = 3$

$X = 15 \times 961.54 + 20 \times 966.14 + 30 \times 878.41$

$= 60{,}098.2$

$$D = \sum_i D_i \frac{X_i}{X}$$

$$= 1 \times \frac{15 \times 961.54}{60{,}098.2} + 2 \times \frac{20 \times 966.14}{60{,}098.2} + 3 \times \frac{30 \times 878.41}{60{,}098.2}$$

$= 2.20$

## Exercise 9

Fund $X$ accumulates at a force of interest of $\delta_t = \dfrac{2}{1+2t}$, where $t$ is measured in years, for $0 \le t \le 20$. Fund $Y$ accumulates at an annual effective interest rate of $i$. An amount of 1 is invested in each fund at time $t = 0$. After 20 years, Fund $X$ has the same value as Fund $Y$.

Calculate the value of Fund $Y$ after five years.

1. 2.46
2. 2.53
3. 2.60
4. 2.67
5. 2.74

$$Y_{20} = 1 \times (1+i)^{20}$$

$$X_{20} = 1 \times e^{\int_0^{20} \delta_t dt} = e^{\int_0^{20} \frac{2}{1+2t} dt} = e^{\ln(1+2t)\Big|_{t=0}^{t=20}}$$

$$= e^{\ln(41)} = 41$$

$$X_{20} = Y_{20} \Rightarrow (1+i)^{20} = 41 \Rightarrow 1+i = \sqrt[20]{41}$$

$$\Rightarrow Y_5 = (1+i)^5 = \left(\sqrt[20]{41}\right)^5 = 2.53$$

# Exercise 10

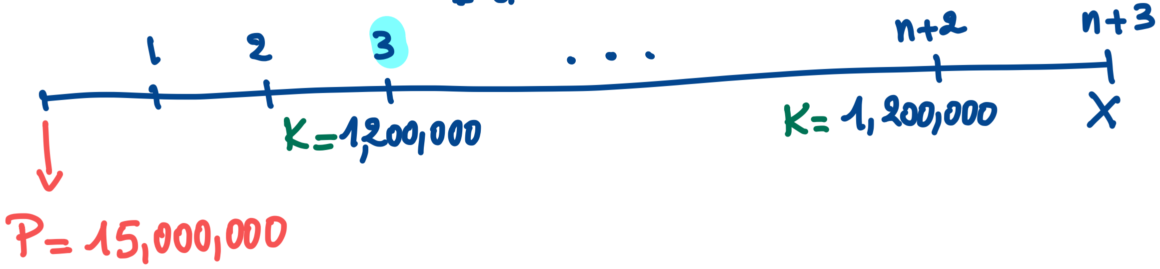
A company takes out of a loan of 15, 000, 000 at an annual effective discount rate of 5.5%. You are given

- The loan is to be repaid with $n$ annual payments of 1, 200, 000 plus a drop payment one year after the $n$-th payment.
- The first payment is due three years after the loan is taken out.

Calculate the amount of the drop payment.

1. 79, 100
2. 176, 000
3. 321, 300
4. 959, 500
5. 1, 180, 300

$$d=5.5\% \Rightarrow i=\frac{1}{1-d}-1=5.82\%$$




$$P > K a_{\overline{n}|i}\, v^2 = K v^2 \frac{1-v^n}{i} \Rightarrow 1-v^n < \frac{iP}{Kv^2} \Rightarrow v^n > 1-\frac{iP}{Kv^2}$$

$$\Rightarrow n \ln v > \ln\left(1-\frac{iP}{Kv^2}\right) \Rightarrow n < \frac{\ln\left(1-\frac{iP}{Kv^2}\right)}{\ln v} = 29.79$$

$$\Rightarrow \boxed{n=29}$$

$$P = 1{,}200{,}000 \times a_{\overline{29}|0.0582} \times 1.0582^{-2} + X \times 1.0582^{-32}$$

$$\Rightarrow X = \frac{P - 1{,}200{,}000 \times a_{\overline{29}|0.0582} \times 1.0582^{-2}}{1.0582^{-32}}$$

$$= 959495$$

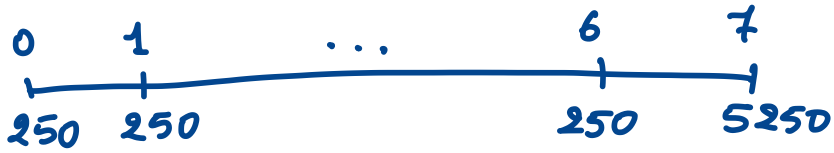
## Exercise 11

Sam buys an eight-year, 5,000 par bond with an annual coupon rate of 5%, paid annually. The bond sells for 5,000. Let $D_1$ be the Macaulay duration just before the first coupon is paid. Let $D_2$ be the Macaulay duration just after the first coupon is paid.

Calculate $\frac{D_1}{D_2}$.

1. 0.91
2. 0.93
3. 0.95
4. 0.97
5. 1.00

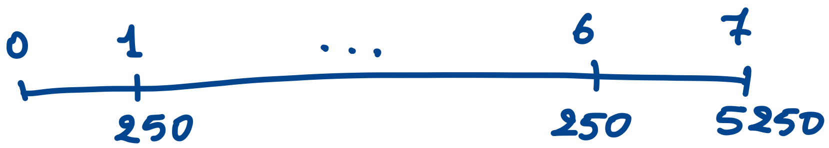
Before the first coupon is paid




It is clear that $j = r$.

$$D_1 = \frac{250\,(Ia)_{\overline{7}|r} + 7 \times v^7 \times 5000}{250\,\ddot{a}_{\overline{8}|r} + 5000\,v^7} = 5.07$$

After the first coupon is paid




It is clear that $j = r$.

$$D_2 = \frac{250\,(Ia)_{\overline{7}|r} + 7 \times v^7 \times 5000}{250\,a_{\overline{7}|r} + 5000\,v^7} = 5.32$$

$$\Rightarrow \frac{D_1}{D_2} = 0.95$$

## Exercise 12

You are given the following information about two 10-year bonds. Both bonds have face amount 100 and coupons payable semi-annually, with next coupon due in 1/2-year.

- Bond 1: Coupon rate 4% per year, price 85.12.
- Bond 2: Coupon rate 10% per year, price 133.34.

Find the yield rate for a 10-year zero coupon bond.

1. 6.06%
2. 6.26%
3. 6.46%
4. 6.66%
5. 6.86%

$F = 100 \; ; \quad n = 20$

*) Bond 1: $r_1 = 2\%$

$$P_1 = \frac{Fr_1}{1+s_0(1)} + \cdots + \frac{Fr_1}{(1+s_0(19))^{19}} + \frac{F+Fr_1}{(1+s_0(20))^{20}}$$

$$= 2\sum_{i=1}^{19} \frac{1}{(1+s_0(i))^i} + \frac{102}{(1+s_0(20))^{20}}$$

*) Bond 2: $r_2 = 5\%$

$$P_2 = \frac{Fr_2}{1+s_0(2)} + \cdots + \frac{Fr_2}{(1+s_0(19))^{19}} + \frac{F+Fr_2}{(1+s_0(20))^{20}}$$

$$= 5\sum_{i=1}^{19} \frac{1}{(1+s_0(i))^i} + \frac{105}{(1+s_0(20))^{20}}$$

$$5P_1 - 2P_2 = \frac{5 \times 102 - 2 \times 105}{(1+s_0(20))^{20}}$$

$$\Rightarrow 158.92 = \frac{300}{(1+s_0(20))^{20}}$$

$$\Rightarrow s_0(20) = \sqrt[20]{\frac{300}{158.92}} - 1 = 3.23\% \text{ per 6 months}$$

As nominal rate, it is equivalent to 6.46%

## Exercise 13

David can receive one of the following two payment streams:

- 100 at time 0, 200 at time $n$ years, and 300 at time $2n$ years.
- 600 at time 10 years.

At an annual effective interest rate of $i$, the present values of the two streams are equal. Given $v^n = 0.76$, calculate $i$.

1. 3.5%
2. 4.0%
3. 4.5%
4. 5.0%
5. 5.5%

$$P_1 = 100 + 200v^n + 300v^{2n} = 100 + 200 \times 0.76 + 300 \times 0.76^2$$
$$= 425.28$$

$$P_2 = 600v^{10}$$

$$P_1 = P_2 \Rightarrow 600v^{10} = 425.28$$
$$\Rightarrow i = \sqrt[10]{\frac{600}{425.28}} - 1$$
$$= 0.035$$

## Exercise 14

- Project $A$ requires an investment of $4,000$ today. The investment pays $2,000$ one year from today and $4,000$ two years from today.
- Project $B$ requires an investment of $X$ two years from today. The investment pays $2,000$ today and $4,000$ one year from today.

The net present values of the two projects are equal at an annual effective interest rate of 10%. Calculate $X$.

1. 5,400
2. 5,420
3. 5,440
4. 5,460
5. 5,480

$$NPV_A = -4000 + 2000v + 4000v^2$$
$$= 1124$$

$$NPV_B = 2000 + 4000v - Xv^2$$

$$NPV_A = NPV_B \Rightarrow X = \frac{2000 + 4000v - 1124}{v^2}$$
$$= 5460$$

## Exercise 15

A perpetuity-immediate pays $X$ per year. Brian receives the first $n$ payments, Colleen receives the next $n$ payments, and a charity receives the remaining payments. Brian's share of the present value of the original perpetuity is 40%, and the charity's share is $K$. Calculate $K$.

1. 24%
2. 28%
3. 32%
4. 36%
5. 40%

$$P = \frac{X}{i} \; ; \; P_{Brian} = X a_{\overline{n}|i} = 0.4P$$
$$= 0.4\frac{X}{i}$$

$$\Rightarrow X\frac{1-v^n}{i} = 0.4\frac{X}{i} \Rightarrow v^n = 0.6$$

$$P_{charity} = \frac{X}{i}v^{2n} = \frac{X}{i} \times 0.6^2 = 0.36P$$

## Exercise 16

An $n$-year bond with annual coupons has the following characteristics:

- The redemption value at maturity is 1890.
- The annual effective yield rate is 6%.
- The book value immediately after the third coupon is 1254.87.
- The book value immediately after the fourth coupon is 1277.38.

Calculate $n$.

1. 16
2. 17
3. 18
4. 19
5. 20

$$C = 1890; \quad j = 6\%$$

$$(1)\ BV_3 = 1254.87 = Fr\, a_{\overline{n-3}|j} + Cv^{n-3} = Fr\left[v + v^2 + \cdots + v^{n-3}\right] + Cv^{n-3}$$

$$(2)\ BV_4 = 1277.38 = Fr\, a_{\overline{n-4}|j} + Cv^{n-4} = Fr\left[v + v^2 + \cdots + v^{n-4}\right] + Cv^{n-4}$$

$$(1) - v \times (2) \Rightarrow 49.79 = Frv \Rightarrow Fr = 52.78$$

Reuse (1)
$$1254.87 = Fr\,\frac{1 - v^{n-3}}{j} + Cv^{n-3}$$
$$= \frac{Fr}{j} + v^{n-3}\left(C - \frac{Fr}{j}\right)$$

$$\Rightarrow v^{n-3} = \frac{1254.87 - \frac{Fr}{j}}{C - \frac{Fr}{j}} = 0.371365$$

$$\Rightarrow n = 3 + \frac{\ln(0.371365)}{\ln(v)} = 20$$

# Exercise 17

You are given the following information with respect to a bond:

- par value: 1000
- term to maturity: 3 years
- annual coupon rate: 6% payable annually

You are also given that the one, two, and three year annual spot interest rates are 7%, 8%, and 9% respectively. Calculate the value of the bond.

1. 906
2. 926
3. 930
4. 950
5. 1000

$$P = \frac{60}{1+s_0(1)} + \frac{60}{(1+s_0(2))^2} + \frac{1060}{(1+s_0(2))^3}$$

$$= 926$$

## Exercise 18

At an annual effective interest rate of 10.9%, each of the following are equal to $X$

- The accumulated value at the end of $n$ years of an $n$-years annuity-immediate paying 21.80 per year.
- The present value of a perpetuity-immediate paying $19,208$ at the end of each $n$-year period.

Calculate $X$.

1. 1555
2. 1750
3. 1960
4. 2174
5. There is not enough information given to calculate $X$

1/


$$X = 21.8\, s_{\overline{n}|0.109} = 21.8 \times \frac{1.109^n - 1}{0.109} \quad (1)$$

2/


$$X = \frac{19,208}{1.109^n - 1} \quad (2)$$

$$(1) = (2) \quad \Rightarrow 21.8\, \frac{1.109^n - 1}{0.109} = \frac{19,208}{1.1.09^n - 1}$$

$$\Rightarrow (1.109^n - 1)^2 = 96.04 \Rightarrow 1.109^n - 1 = \sqrt{96.04}$$

$$(2) \Rightarrow X = \frac{19208}{\sqrt{96.04}} = 1960$$

## Exercise 19

An investor decides to purchase a five-year annuity with an annual nominal interest rate of 12% convertible monthly for a price of $X$.

Under the terms of the annuity, the investor is to receive 2 at the end of the first month. The payments increase by 2 each month thereafter.

Calculate $X$.

1. 2015
2. 2386
3. 2475
4. 2500
5. 2524

$i = 1\%$ ; $n = 5 \times 12 = 60$ ; $K = 2$

$$X = 2\,(Ia)_{\overline{60}|0.01}$$

$$= 2 \times \frac{\ddot{a}_{\overline{60}|0.01} - 60 \times 1.01^{-60}}{0.01}$$

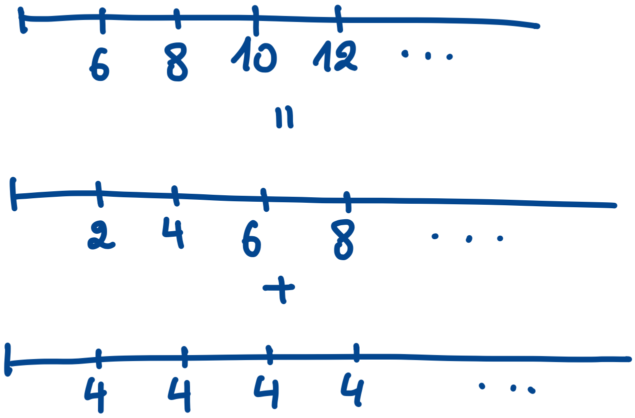
$$= 2475.52\,\$$$

## Exercise 20

At an annual effective interest rate of 6%, the present value of a perpetuity immediate with successive annual payments of $6, 8, 10, 12, 14, \ldots$, is equal to $X$. Calculate $X$.

1. 100
2. 656
3. 695
4. 1767
5. 1793

$i = 6\%$




$$X = 2(Ia)_{\overline{\infty}|0.06} + \frac{4}{0.06}$$

$$= 2 \times \left[\frac{1}{0.06} + \frac{1}{0.06^2}\right] + \frac{4}{0.06}$$

$$= 655.55$$

## Exercise 21

A railroad company is required to pay 79,860, which is due three years from now. The company invests 15,000 in a bond with modified duration 1.80, and 45,000 in a bond with modified duration $X$, to Redington immunize its position against small changes in the yield rate.

The annual effective yield rate for each of the bonds is 10%. Calculate $X$.

1. 2.73
2. 3.04
3. 3.34
4. 3.40
5. 3.65

$i = 10\%$

$L_3 = 79{,}860 \Rightarrow PV_L = L_3 v^3 = 60{,}000$

$$DM_L = -\frac{\frac{dPV_L}{di}}{PV_L} = \frac{L_3 \times 3 \times v^4}{L_3 v^3} = 3v$$

$$DM_A = \frac{15{,}000}{60{,}000} \times 1.8 + \frac{45{,}000}{60{,}000} \times X$$

$$DM_L = DM_A \Rightarrow \frac{3}{4} X = 3v - \frac{1}{4} \times 1.8$$

$$\Rightarrow X = 4v - \frac{1.8}{3}$$

$$= 4 \times 1.1^{-1} - \frac{1.8}{3}$$

$$= 3.036$$

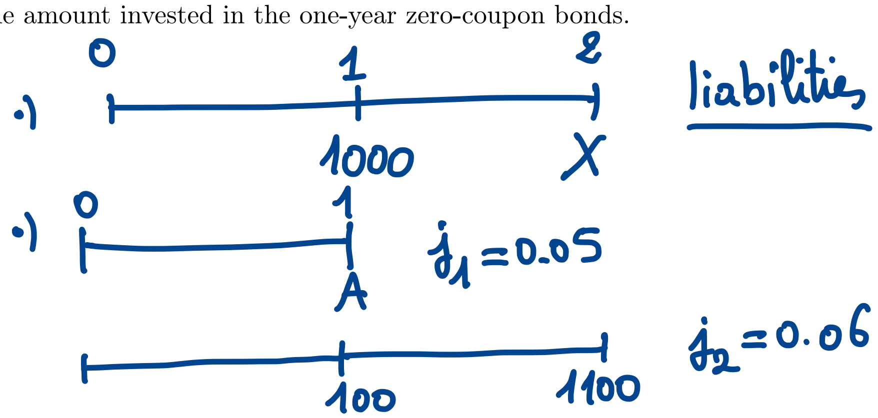
## Exercise 22

A company must pay liabilities of 1,000 at the end of year 1 and $X$ at the end of year 2. The only investments available are:

- One-year zero-coupon bonds with an annual effective yield of 5%.
- Two-year bonds with a par value of 1,000 and 10% annual coupons, with an annual effective yield of 6%.

The company constructed a portfolio that creates an exact cash flow matching strategy for these liabilities. The total purchase price of this portfolio is 1783.76. Calculate the amount invested in the one-year zero-coupon bonds.

1. 784
2. 831
3. 871
4. 915
5. 935




Let A be the redemption value of the one-year zero-coupon bond and B be the number of two-year bonds

We have $1783.76 = \frac{A}{1+0.05} + B\left[\frac{100}{1.06} + \frac{1100}{1.06^2}\right]$

$\Rightarrow 0.9524A + 1073.3357B = 1783.76 \quad (1)$

Also, we have $100B + A = 1000 \quad (2)$

$(2) \times 1073.3357 - 100 \times (1)$

$\Rightarrow 978.0957A = 894959.7$

$\Rightarrow A = 915 \Rightarrow$ The amount invested in one-year zero-coupon is $\frac{A}{1.06} = 871.43\$$

# Exercise 23

A bond has a modified duration of 8 and a price of $112,955$ calculated using an annual effective interest rate of 6.4%.

$E_{MAC}$ is the estimated price of this bond at an interest rate of 7.0% using the first-order Macaulay approximation.

$E_{MOD}$ is the estimated price of this bond at an interest rate of 7.0% using the first-order modified approximation.

Calculate $E_{MAC} - E_{MOD}$.

1. 102
2. 116
3. 127
4. 143
5. 453

$P_0 = 112,955 \ ; \ i_0 = 6.4\% \ ; \quad DM = 8$

$\Rightarrow D = (1+i_0) \times DM$

$$E_{Mac} = P_0 \times \left(\frac{1+i_0}{1+i}\right)^D$$

$$= 112,955 \times \left(\frac{1.064}{1.07}\right)^{1.064 \times 8}$$

$$= 107,675.74$$

$$E_{Mod} = P_0 \left[1 - h \times DM\right]$$

$$= 112,955 \times \left[1 - (0.07 - 0.064) \times 8\right]$$

$$= 107,533.16$$

$\Rightarrow E_{Mac} - E_{Mod} = 142.58\$$

## Exercise 24

An investor deposits 100 into a bank account at time 0. The bank credits interest at an annual nominal interest rate of $i$, compounded semi-annually. The total amount of interest credited in the twelfth year is twice the amount of interest credited in the fifth year. Calculate $i$.

1. 10.15%
2. 10.24%
3. 10.32%
4. 10.41%
5. 10.48%

Let $j$ be the interest per 6 months. Then $j = \frac{i}{2}$.

The interest credited in the fifth year is

$$100\left[(1+j)^{10} - (1+j)^{8}\right] = 100(1+j)^{8}\left[(1+j)^{2} - 1\right]$$

The interest credited in the twelfth year is

$$100\left[(1+j)^{24} - (1+j)^{22}\right] = 100(1+j)^{22}\left[(1+j)^{2} - 1\right]$$

$$100(1+j)^{22}\left[(1+j)^{2} - 1\right] = 2 \times 100 \times (1+j)^{8}\left[(1+j)^{2} - 1\right]$$

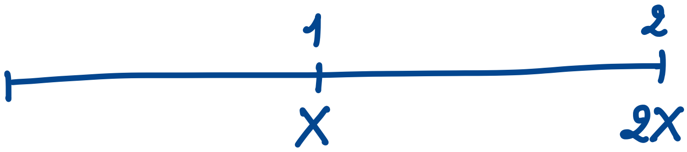
$$\Rightarrow (1+j)^{14} = 2 \Rightarrow j = \sqrt[14]{2} - 1$$

$$\Rightarrow i = 2j = 2\left[\sqrt[14]{2} - 1\right] = 0.1015$$

## Exercise 25

A two-year loan of 100 is repaid with a payment of $X$ at the end of the first year and $2X$ at the end of the second year. The annual effective interest rate charged by the lender is 8% in the first year and $i$ in the second year. The annual effective yield rate for the lender is 10%. Calculate $i$.

1. 12.8%
2. 12.9%
3. 13.0%
4. 13.1%
5. 13.2%




$$\begin{cases} 100 = X \times 1.08^{-1} + 2X \times 1.08^{-1} \times \dfrac{1}{1+i} & (1) \\ 100 = X \times 1.1^{-1} + 2X \times 1.1^{-2} & (2) \end{cases}$$

$$(2) \Rightarrow X = \frac{100}{1.1^{-1} + 2 \times 1.1^{-2}} = 39.03$$

$$(1) \Rightarrow i = \frac{2X \times 1.08^{-1}}{100 - X \times 1.08^{-1}} - 1$$

$$= 0.1318$$

## Exercise 26

A mortgage for 125,000 has level payments at the end of each month and an annual nominal interest rate compounded monthly. The balances owed immediately after the first and second payments were 124,750 and 124,498, respectively.

Calculate the number of payments needed to pay off the mortgage.

1. 198
2. 199
3. 200
4. 201
5. 202

$$PR_1 = Kv^n = 250 \quad (1)$$

$$PR_2 = Kv^{n-1} = 252 \quad (2)$$

$$(1) \div (2) \Rightarrow v = \frac{250}{252} \Rightarrow i = \frac{252}{250} - 1$$

$$125{,}000 = K\, a_{\overline{n}|i} = K \cdot \frac{1-v^n}{i} = \frac{K - Kv^n}{i}$$

$$= \frac{K - 250}{i} \Rightarrow K = i \times 125{,}000 + 250$$

$$= 1250$$

Reusing (1) $\Rightarrow$ $v^n = \dfrac{250}{K}$

$$\Rightarrow n = \frac{\ln\left(\frac{250}{K}\right)}{\ln v} = \frac{\ln\left(\frac{250}{1250}\right)}{\ln\left(\frac{250}{252}\right)}$$

$$\simeq 202$$

## Exercise 27

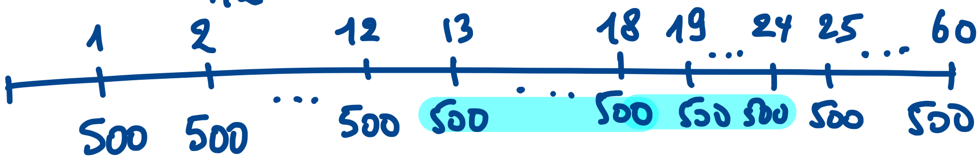
A five-year loan has an annual nominal interest rate of 30% convertible monthly. The loan is scheduled to be repaid with level monthly payments of 500, beginning one month after the date of the loan.

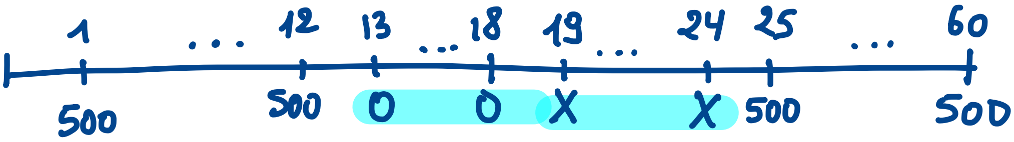
The borrower misses the thirteenth through the eighteenth payments, but increases the next six payments to $X$ so that the final 36 payments of 500 will repay the loan.

Calculate $X$.

1. 1065
2. 1070
3. 1075
4. 1080
5. 1085

$i = \frac{30\%}{12} = 2.5\%$ ; $K = 500$ ; $n = 5 \times 12 = 60$



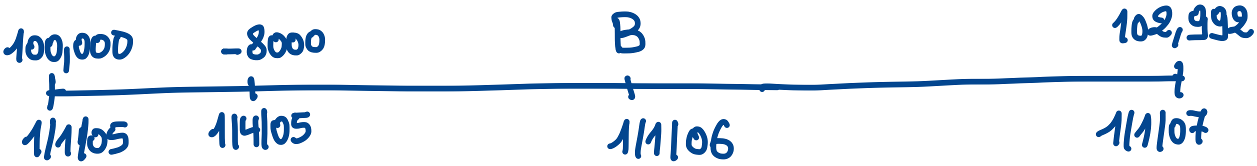

$$500\, a_{\overline{12}|0.025} \times v^{12} = X\, a_{\overline{6}|0.025}\, v^{18}$$

$$\Rightarrow X = \frac{500\, a_{\overline{12}|0.025}}{a_{\overline{6}|0.025} \times v^{6}} = 1079.84$$

## Exercise 28

On January 1, 2005, an investment account is worth 100,000. On April 1, 2005, the value has increased to 103,000 and 8,000 is withdrawn. On January 1, 2007, the account is worth 102,992. Assuming a dollar-weighted method for 2005 and a time-weighted method for 2006, the effective annual interest rate was equal to $x$ for both 2005 and 2006. Calculate $x$.

1. 5.74%
2. 5.96%
3. 6.14%
4. 6.38%
5. 6.55%




**2005:** $$x = \frac{B - [100{,}000 - 8000]}{100{,}000 - 8000 \times \frac{3}{4}} = \frac{B - 92{,}000}{94{,}000}$$

**2006:** $$x = \frac{102{,}992}{B} - 1 = \frac{102{,}992 - B}{B}$$

$$\frac{B - 92{,}000}{94{,}000} = \frac{102{,}992 - B}{B}$$

$$\Rightarrow B^2 - 92{,}000B = 102{,}992 \times 94{,}000 - 94{,}000B$$

$$\Rightarrow B = 97398.41$$

$$\Rightarrow x = \frac{102{,}992}{B} - 1 = 0.0574$$

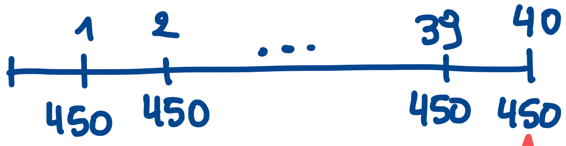
## Exercise 29

Jerry will make deposits of 450 at the end of each quarter for 10 years. At the end of 15 years, Jerry will use the fund to make annual payments of $Y$ at the beginning of each year for 4 years, after which the fund is exhausted. The annual effective rate of interest is 7%.

Determine $Y$.

1. 9473
2. 9573
3. 9673
4. 9773
5. 9873

$K=450$; $n=10\times 4=40$; $j=\sqrt[4]{1.07}-1=0.01706$

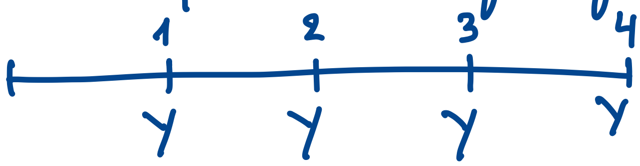



$S=450\, s_{\overline{40}|j}=25511.$

After another 5 years, the fund becomes

$$S\times(1+0.07)^5=35{,}780.53\$$$

This is the present value of the following annuity




$$35{,}780.53=Y\,\ddot{a}_{\overline{4}|0.07}$$

$$\Rightarrow Y=\frac{35{,}780.53}{\ddot{a}_{\overline{4}|0.07}}=9872.35\$$$

## Exercise 30

Mary purchased a 10-year par value bond with an annual nominal coupon rate of 4% payable semiannually at a price of 1021.50. The bond can be called at 100 over the par value of 1100 on any coupon date starting at the end of year 5 and ending six months prior to maturity.

Calculate the minimum yield that Mary could receive, expressed as an annual nominal rate of interest convertible semiannually.

1. 4.7%
2. 4.9%
3. 5.1%
4. 5.3%
5. 5.5%

$r = 2\%$ ; 20 payments ; $P = 1021.5$

$F = 1100 \Rightarrow Fr = 22$

For $n \in \{10, 11, \dots, 19\} \Rightarrow C = 1200$

Given that $P < C$, then the minimum yield rate is calculated based on a call at the last possible date

$\Rightarrow P = Cv^{19} + Fr\, a_{\overline{19}|j} = 1222\, v^{19} + 22\, a_{\overline{18}|j}$

$\Rightarrow j = 2.8589\% \Rightarrow (5.7178\%)$

For $n = 20 \Rightarrow C = 1100$

$P = 1122\, v^{20} + 22\, a_{\overline{19}|j} \Rightarrow j = 2.4559\%$

$(4.9118\%)$

1. With force of interest $\delta_t$:

$$A(t_2) = A(t_1)e^{\int_{t_1}^{t_2} \delta_s ds}$$

2. Annuity immediate

$$a_{\overline{n|}\ i} = \frac{1-(1+i)^{-n}}{i} \ \ ; \ \ s_{\overline{n|}\ i} = \frac{(1+i)^n - 1}{i}$$

3. Annuity due

$$\ddot{a}_{\overline{n|}\ i} = \frac{1-v^n}{d} \ \ ; \ \ \ddot{s}_{\overline{n|}\ i} = \frac{(1+i)^n - 1}{d} \ \ ; \ \ d = \frac{i}{1+i}$$

4. Geometric annuity immediate

$$PV = \frac{1-\left(\frac{1+r}{1+i}\right)^n}{i-r} \ \ ; \ \ FV = \frac{(1+i)^n - (1+r)^n}{i-r}$$

5. Increasing arithmetic annuity immediate

$$(Ia)_{\overline{n|}\ i} = \frac{\ddot{a}_{\overline{n|}\ i} - nv^n}{i} \ \ ; \ \ (Is)_{\overline{n|}\ i} = \frac{\ddot{s}_{\overline{n|}\ i} - n}{i}$$

6. Decreasing arithmetic annuity immediate

$$(Da)_{\overline{n|}\ i} = \frac{n - a_{\overline{n|}\ i}}{i} \ \ ; \ \ (Ds)_{\overline{n|}\ i} = \frac{n(1+i)^n - s_{\overline{n|}\ i}}{i}$$

7. Present value of perpetuity immediate

$$a_{\overline{\infty|}\ i} = \frac{1}{i}$$

8. Present value of perpetuity due

$$\ddot{a}_{\overline{\infty|}\ i} = \frac{1+i}{i}$$

9. Present value of geometric perpetuity immediate (if $r < i$)

$$PV = \frac{1}{i-r}$$

10. Present value of increasing arithmetic perpetuity immediate

$$(Ia)_{\overline{\infty|}\ i} = \frac{1}{i} + \frac{1}{i^2}$$

11. Loan amortization

$$OB_t = OB_{t-1}(1+i) - K_t \ \ ; \ \ I_t = OB_{t-1} \times i \ \ ; \ \ PR_t = K_t - I_t.$$

12. Retrospective form

$$OB_t = OB_0(1+i)^t - K_1(1+i)^{t-1} - K_2(1+i)^{t-2} - \cdots - K_{t-1}(1+i) - K_t$$

13. Prospective form

$$OB_t = K_{t+1} \times v + K_{t+2} \times v^2 + \cdots + K_n \times v^{n-t}$$

14. Amortization with $n$ level payment of amount $K$ each

$$OB_t = Ka_{\overline{n-t|}\ i} \ \ ; \ \ I_t = K(1 - v^{n-t+1}) \ \ ; \ \ PR_t = Kv^{n-t+1}$$

15. Bond price

$$P = C + (Fr - Cj)a_{\overline{n|}\ j}$$

16. Book value

$$BV_{t+1} = BV_t \times (1 + j) - Fr \ \ ; \ \ I_{t+1} = BV_t \times j \ \ ; \ \ PR_{t+1} = Fr - I_{t+1}$$

17. Internal Rate of Return $i$ is the solution of the equation

$$\sum_k C_k v^{t_k} = 0$$

18. Dollar weighted return for a one-year period is

$$\frac{B - [A + \sum_{k=1}^{n} C_k]}{A + \sum_{k=1}^{n} C_k(1 - t_k)}$$

19. Time-weighted return for a one-year period

$$\frac{F_1}{A} \times \frac{F_2}{F_1 + C_1} \times \cdots \times \frac{F_n}{F_{n-1} + C_{n-1}} \times \frac{B}{F_n + C_n} - 1$$

20.

$$DM = \frac{\sum_t tC_t v^{t+1}}{\sum_t C_t v^t} \ \ ; \ \ D = \frac{\sum_t tC_t v^t}{\sum_t C_t v^t} \ \ ; \ \ C_{mod} = \frac{\sum_t t(t+1)C_t v^{t+2}}{\sum_t C_t v^t} \ \ ; \ \ C_{mac} = \frac{\sum_t t^2 C_t v^t}{\sum_t C_t v^t}$$

21.

$$P(i) \approx P(i_0)(1 - h \times DM) \ \ ; \ \ P(i) \approx P(i_0)\left(\frac{1 + i_0}{1 + i}\right)^D$$