# AS289- Major Exam 1 (Code 1)

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## Exercise 1

How much more interest (in dollars) is earned on an investment today of $\$1,000$ during the 4-th year assuming effective annual compound interest of 5% rather than assuming annual simple interest of 5%

1. 0
2. 4.3
3. 7.9
4. 10.8
5. 12.2

Scenario 1: simple interest of 5%

Interest amount is $0.05 \times 1000 = 50\,\$$

Scenario 2: effective annual compound interest of 5%

Interest amount is

$$1000(1+0.05)^4 - 1000(1+0.05)^3$$
$$= 1000 \times 1.05^3 \times 0.05$$
$$\simeq 57.9\,\$$$

Extra is $7.9\,\$$

## Exercise 2

A bank customer takes out a loan of 500 with a 16% nominal interest rate convertible quarterly. The customer makes payments of 20 at the end of each quarter.

Calculate the amount of principal in the fourth payment.

1. 0.0
2. 0.9
3. 2.7
4. 5.2
5. There is not enough information to calculate the amount of principal

$i^{(4)} = 16\%$ ; $L = 500$ ; $K = 20$

Note that $L \frac{i^{(4)}}{4} = 500 \times 0.04 = 20$.

Then, each payment will cover only the interest.

## Exercise 3

The parents of three children, ages 1, 3, and 6, wish to set up a trust fund that will pay $X$ to each child upon attainment of age 18, and $Y$ to each child upon attainment of age 21. They will establish the trust fund with a single investment of $Z$.

Which of the following is the correct equation of value for $Z$?

1. $\dfrac{X}{v^{12}+v^{15}+v^{17}}+\dfrac{Y}{v^{15}+v^{18}+v^{20}}$
2. $3[Xv^{18}+Yv^{21}]$
3. $3Xv^{3}+Y[v^{15}+v^{18}+v^{20}]$
4. $(X+Y)\dfrac{v^{15}+v^{18}+v^{20}}{v^{3}}$
5. $X[v^{12}+v^{15}+v^{17}]+Y[v^{15}+v^{18}+v^{20}]$

It is clear that $Z = X\left[v^{17}+v^{15}+v^{12}\right]+Y\left[v^{20}+v^{18}+v^{15}\right]$

## Exercise 4

At a nominal interest rate of $i$ convertible semi-annually, an investment of 1000 immediately and 1500 at the end of the first year will accumulated to 2600 at the end of the second year. Calculate $i$.

1. 2.75%
2. 2.77%
3. 2.79%
4. 2.81%
5. 2.83%

$i^{(2)} = i$

$1000 \times \left(1+\frac{i}{2}\right)^4 + 1500 \times \left(1+\frac{i}{2}\right)^2 = 2600$

Let $X = \left(1+\frac{i}{2}\right)^2$.

$1000X^2 + 1500X - 2600 = 0$

$\Rightarrow 10X^2 + 15X - 26 = 0$

$\Delta = 15^2 + 4 \times 10 \times 26 = 1265$

$\Rightarrow \begin{cases} X_1 = \dfrac{-15-\sqrt{1265}}{2\times 10} \text{ to be rejected} \\ X_2 = \dfrac{-15+\sqrt{1265}}{2\times 10} = 1.028342 \end{cases}$

$\left(1+\frac{i}{2}\right)^2 = X_2 \Rightarrow i = 2\left(\sqrt{X_2}-1\right) = 0.028144$

## Exercise 5

Bruce and Robbie each open up new bank accounts at time 0. Bruce deposits 100 into his bank account, and Robbie deposits 50 into her bank account. Each account earns the same annual effective interest rate.

The amount of interest earned in Bruce's account during the 11-th year is equal $X$. The amount of interest earned in Robbie's account during the 17-th year is also equal to $X$.

Calculate $X$.

1. 28.00
2. 31.30
3. 34.60
4. 36.70
5. 38.90

Bruce:


$$X = 100(1+i)^{11} - 100 \times (1+i)^{10} = 100\, i\, (1+i)^{10}$$

Robbie:


$$X = 50(1+i)^{17} - 50 \times (1+i)^{16} = 50\, i\, (1+i)^{16}$$

$$100\, i\, (1+i)^{10} = 50\, i\, (1+i)^{16} \Longrightarrow 2 = (1+i)^6 \Rightarrow i = \sqrt[6]{2} - 1$$
$$= 0.12246$$

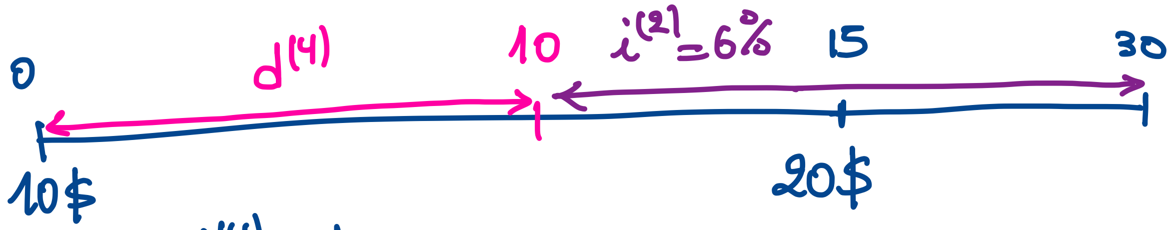
$$\Longrightarrow X = 50\, i\, (1+i)^{16} \simeq 38.9$$

## Exercise 6

Jeff deposits 10 into a fund today and 20 fifteen years later. Interest is credited at a nominal discount rate of $d$ compounded quarterly for the first 10 years, and at a nominal interest rate of 6% compounded semiannually thereafter. The accumulated balance in the fund at the end of 30 years is 100.

Calculate $d$.

1. 4.33%
2. 4.43%
3. 4.53%
4. 4.63%
5. 4.73%




$d^{(4)} = d$ ;

$$\left[100 \times \left(1+\frac{0.06}{2}\right)^{-30} - 20\right] \times \left(1+\frac{0.06}{2}\right)^{-10} \times \left(1-\frac{d}{4}\right)^{40} = 10$$

$$\Rightarrow \left(1-\frac{d}{4}\right)^{40} \approx 0.63396241$$

$$\Rightarrow d \approx 4\left(1-\sqrt[40]{0.63396241}\right)$$

$$\approx 0.0453$$

# Exercise 7

An insurance company has an obligation to pay the medical costs for a claimant. Average annual claims costs today are \$5,000, and medical inflation is expected to be 7% per year. The claimant is expected to live an additional 20 years. Claim payments are made at yearly intervals, with the first claim payment to be made one year from today.

Find the present value of the obligation in the annual interest rate is 5%.

1. 87,932
2. 102,514
3. 114,611
4. 122,634
5. Cannot be determined

1    2    . . .    20

$5000 \times (1+r)$    $5000(1+r)^2$    $5000(1+r)^{20}$

where $r = 7\%$

$i = 5\%$

$$PV = 5000 \times (1+r) \times \frac{1 - \left(\frac{1+r}{1+i}\right)^{20}}{i-r}$$

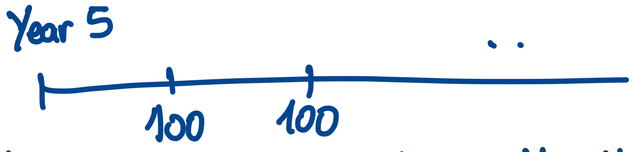
$= 122,633.6$ \$

# Exercise 8

A perpetuity-immediate pays 100 per year. Immediately after the fifth payment, the perpetuity is exchanged for a 25-year annuity-immediate that will pay $X$ at the end of the first year. Each subsequent annual payment will be 8% greater than the preceding payment. The annual effective rate of interest is 8%.

Calculate $X$

1. 54
2. 64
3. 74
4. 84
5. 94




The value of the perpetuity after the fifth payment is

$$PV = \frac{100}{i} = 1250$$

This is exactly the present value of the annuity


$$PV = Xv + Xv^2(1+r) + \dots + Xv^{25}(1+r)^{24}$$

$$= Xv + Xv + \dots + Xv$$ (as $r = i \Rightarrow v(1+r) = 1$)

$$= 25Xv$$

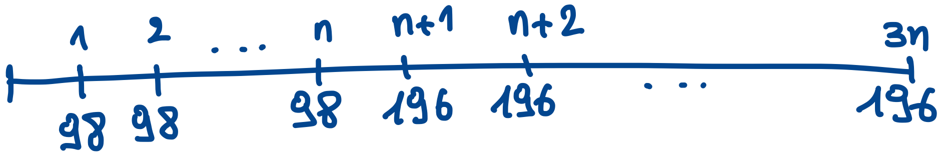
$$\Rightarrow X = \frac{PV}{25v} = \frac{PV(1+i)}{25} = 54$$

# Exercise 9

To accumulate 8,000 at the end of $3n$ years, deposits of 98 are made at the end of each of the first $n$ years and 196 at the end of each of the next $2n$ years. The annual effective rate of interest is $i$. You are given $(1+i)^n = 2$.

Determine $i$

1. 11.25%
2. 11.75%
3. 12.25%
4. 12.75%
5. 13.25%




$$98\, s_{\overline{n}|i}\,(1+i)^{2n} + 196\, s_{\overline{2n}|i} = 8000$$

$$\Rightarrow 98 \times \frac{(1+i)^n - 1}{i} \times (1+i)^{2n} + 196 \times \frac{(1+i)^{2n} - 1}{i} = 8000$$

$$\Rightarrow 98 \times \frac{4}{i} + 196 \times \frac{3}{i} = 8000$$

$$\Rightarrow \frac{980}{i} = 8000 \Rightarrow i = 0.1225$$

## Exercise 10

Which of the following are characteristics of all perpetuities?

- (I) The present value is equal to the first payment divided by the annual effective interest rate.. false, does not cover geometric or arithmetic perpetuity
- (II) Payments continue forever. True
- (III) Each payment is equal to the interest earned on the principal. false (see the explanation below)

1. (I) only
2. (II) only
3. (III) only
4. (I), (II) and (III)
5. The correct answer is not given above

Suppose a geometric perpetuity of 1\$ with $i = 7\%$ and $r = 5\%$

$$PV = \frac{1}{i-r} = 50\$$$

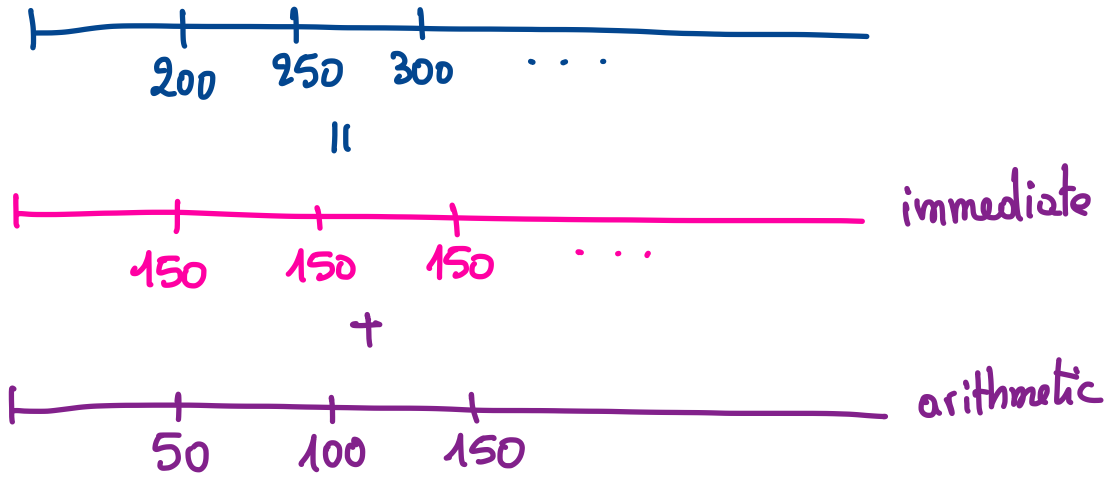
The interest the first year is $PV\,i = 3.5\$$ while the payment is 1\$

## Exercise 11

At an annual effective interest rate $i$, the present value of a perpetuity-immediate starting with a payment of 200 in the first year and increasing by 50 each year thereafter is $46,530$.

Calculate $i$.

1. 3.25%
2. 3.50%
3. 3.75%
4. 4.00%
5. 4.25%




$$46,530 = \frac{150}{i} + 50 \times \left[\frac{1}{i} + \frac{1}{i^2}\right] = \frac{200}{i} + \frac{50}{i^2}$$

$$\Rightarrow 46,530 i^2 - 200 i - 50 = 0$$

$$\Rightarrow 4653 i^2 - 20 i - 5 = 0$$

$$\Delta = (-20)^2 - 4 \times (-5) \times 4653 = 93,460$$

$$i_1 = \frac{20 - \sqrt{93,460}}{2 \times 4653} < 0$$ to be rejected

$$i_2 = \frac{20 + \sqrt{93,460}}{2 \times 4653} = 0.035$$

## Exercise 12

The present value of a 25-year annuity-immediate with a first payment of 2500 and decreasing by 100 each year thereafter is $X$. Assuming an annual effective rate of 10%, calculate $X$.

1. 11, 346
2. 13, 515
3. 15, 923
4. 17, 396
5. 18, 112


$$X = 100\,(Da)_{\overline{25}|0.1}$$

$$= 100 \times \frac{25 - a_{\overline{25}|0.1}}{0.1}$$

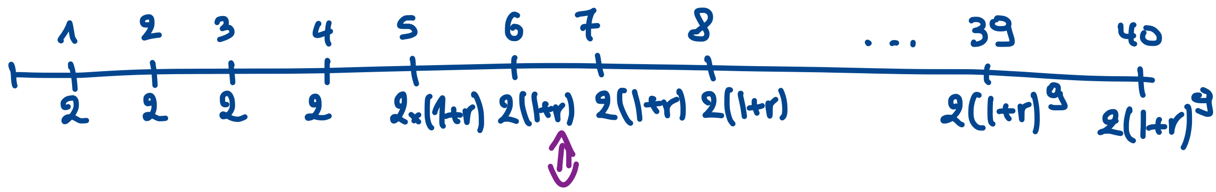
$$= 15{,}923$$

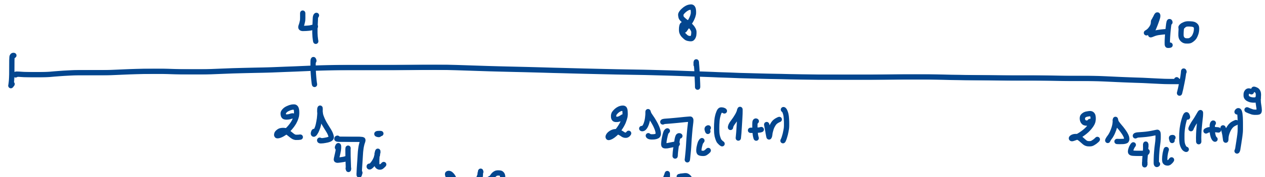
## Exercise 13

An annuity due pays an initial benefit of 2 per year, with the benefit increasing by 5.25% every four years. The annuity is payable for 40 payments.

Using an annual effective rate of 3%, calculate the future value of this annuity immediately after the last payment.

1. 58
2. 76
3. 125
4. 183
5. 197




$$FV = 2s_{\overline{4}|i} \times \frac{\left((1+i)^4\right)^{10} - (1+r)^{10}}{\left(\underbrace{(1+i)^4-1}_{\text{effective interest rate each four year}}\right) - r}$$

$$\simeq 182.67$$

## Exercise 14

A 300,000 home loan is amortized by equal monthly payments for 25 years, starting one month from the time of the loan at a nominal rate of 7% convertible monthly. Which of the following is closest to the total interest paid during the last 10 years of the loan?

1. 71, 820
2. 71, 910
3. 72, 530
4. 72, 660
5. 77, 050

$n = 25 \times 12 = 300$ ; $L = 300{,}000$

$i = \frac{7}{12}$

$$OB_{180} = K a_{\overline{120}|i} = \frac{L}{a_{\overline{300}|i}} \times a_{\overline{120}|i}$$

During the last 10 years, the total principal repaid is $OB_{180}$, while the total payments is $120K = 120 \frac{L}{a_{\overline{300}|i}}$.

Then, the total interest paid during this period is

$$120K - OB_{180} = \frac{L}{a_{\overline{300}|i}} \times \left[120 - a_{\overline{120}|i}\right]$$

$$= 71{,}823.56$$

## Exercise 15

Iggy borrows $X$ for 10 years at an effective annual rate of 6%. If he pays the principal and accumulated interest in one lump sum at the end of 10 years, he would pay 356.54 more in interest than if he repaid the loan with 10 level payments at the end of each year. Calculate $X$.

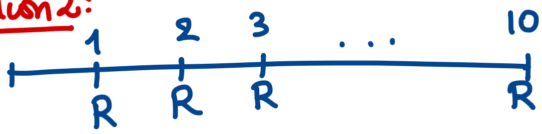
1. 795
2. 805
3. 815
4. 825
5. 835

$OB_0 = X; \quad i = 6\%$

Option 1:

Total $X(1+i)^{10} = 1.06^{10} X$

Option 2:




$OB_0 = X = R\, a_{\overline{10}|0.06} \Rightarrow R = \dfrac{X}{a_{\overline{10}|0.06}}$

Total $10R = \dfrac{10}{a_{\overline{10}|0.06}} X$

$1.06^{10} X = \dfrac{10}{a_{\overline{10}|0.06}} X + 356.54$

$X\left[1.06^{10} - \dfrac{10}{a_{\overline{10}|0.06}}\right] = 356.54$

$\Rightarrow X = \dfrac{356.54}{1.06^{10} - \dfrac{10}{a_{\overline{10}|0.06}}} = 825$

## Exercise 16

A loan is repaid with level annual payments based on an annual effective interest rate of 7%. The 8th payment consists of 789 of interest and 211 of principal.

Calculate the amount of interest paid in the 18th payment

1. 415
2. 444
3. 556
4. 585
5. 612

$i = 7\%$ ; $K = 789 + 211 = 1000\,\$$

$PR_8 = K v^{n-7} = 211$

$\Rightarrow I_{18} = K(1 - v^{n-17})$

$= K - v^{-10} \underbrace{K v^{n-7}}_{211}$

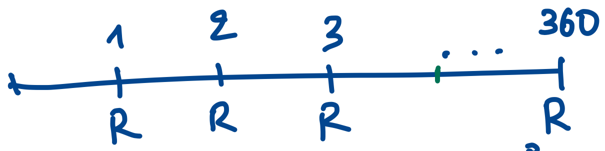
$= 1000 - 1.07^{10} \times 211$

$= 585$

## Exercise 17

A homebuyer borrows 250,000 to be repaid over a 30-year period with level monthly payments beginning one month after the loan is made. The interest rate on the loan is a nominal annual rate of 9% compounded monthly. Find the amount of interest paid in the 30-th year.

1. 997
2. 1025
3. 1078
4. 1106
5. 1137




$OB_0 = 250{,}000$ ; $i = \frac{9\%}{12} = 0.75\%$

$$OB_0 = R\, a_{\overline{360}|0.0075} \Rightarrow R = \frac{OB_0}{a_{\overline{360}|0.0075}} = 2011.55$$

$$OB_{348} = R\, a_{\overline{12}|0.0075} = 23001.89$$


Amount of interest in the 30-th year is

$$12R - OB_{348} = 1136.71$$

## Exercise 18

Hank purchases a \$200,000 home. Mortgage payments are to be made monthly for 30 years with the first payment to be made one month from the loan origination. The annual effective rate of interest is 5%.

Starting with the 100th payment, 400 is added to each payment in order to repay the mortgage earlier. What would be the amount of the last payment.

1. 565
2. 567
3. 1020
4. 1060
5. 1460


$$K = \frac{L}{a_{\overline{360}|i}} \quad \text{where} \quad i = \sqrt[12]{1.05} - 1 \simeq 0.40741\%$$

$$\Rightarrow K = 1{,}060.11\,\$$$

$$OB_{99} = K a_{\overline{261}|i} = 170{,}162.81\,\$$$

Now, we use a new $K' = K + 400 = 1460.11\,\$$

$$169{,}796 = K' a_{\overline{n}|i} = K' \frac{1 - v^n}{i} \Rightarrow n = \frac{\ln\left(1 - \frac{i \times 170{,}162.81}{K'}\right)}{\ln v}$$

$$\simeq 158.39$$

We have 158 regular payments of 1460.11 + a final payment $x$ ?


$$OB_{99} = K' a_{\overline{158}|i} + x v^{159}$$

$$\Rightarrow \quad x = \left(OB_{99} - K' a_{\overline{158}|i}\right)(1+i)^{159} = 567.12\,\$$$

## Exercise 19

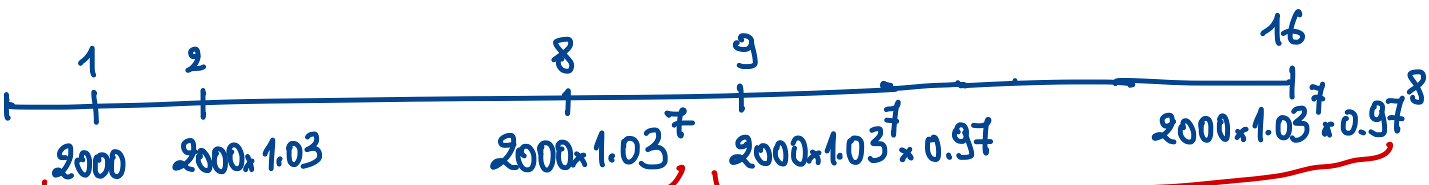
You are given the following information about a loan of $L$ that is to be repaid with a series of 16 annual payments:

- The first payment of 2000 is due one year from now.
- The next seven payments are each 3% larger than the preceding payment.
- From the 9th to the 16th payment, each payment will be 3% less than the preceding payment.
- The loan has an annual effective interest rate of 7%.

Calculate $L$.

1. 20,689
2. 20,716
3. 20,775
4. 21,147
5. 22,137

$i = 0.07$




Geometric with $r_1 = 0.03$
$R = 2000$

Geometric with $r_2 = -0.03$
and $R = 2000 \times 1.03^7 \times 0.97$

$$L = 2000 \times \frac{1-\left(\frac{1+r_1}{1+i}\right)^8}{i-r_1} + v^8 \times 2000 \times 1.03^7 \times 0.97 \times \frac{1-\left(\frac{1+r_2}{1+i}\right)^8}{i-r_2}$$

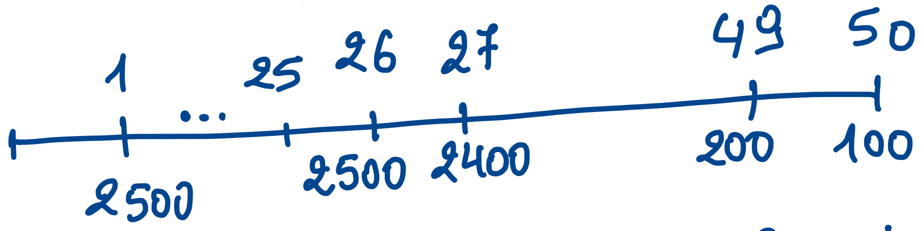
$$= 20688.63$$

## Exercise 20

A borrower takes out a 50-year loan, to be repaid with payments at the end of each year. The loan payment is 2500 for each of the first 26 years. Thereafter, the payments decrease by 100 per year. Interest on the loan is charged at an annual effective rate of $i$ ($0\% < i < 10\%$). The principal repaid in year 26 is $X$.

Determine the amount of interest paid in the first year.

1. $Xv^{25}$
2. $2500v^{25} - Xv^{25}$
3. $2500 - X$
4. $2500 - Xv^{25}$
5. $25Xi$




$$OB_{25} = 100(Da)_{\overline{25}|i} = 100 \times \frac{25 - a_{\overline{25}|i}}{i}$$

$$X = 2500 - OB_{25}\, i = 100\, a_{\overline{25}|i}$$

$$OB_0 = 2500\, a_{\overline{25}|i} + 100(Da)_{\overline{25}|i} \times v^{25}$$

$$= 2500\, a_{\overline{25}|i} + v^{25}\, OB_{25}$$

$$\Rightarrow I_1 = OB_0\, i$$

$$= 2500(1 - v^{25}) + v^{25}(2500 - X)$$

$$= 2500 - v^{25}X$$