AS289- Final

KFUPM, Department of Mathematics

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Exercise 1

Calculate the Macaulay duration of a common stock that pays dividends at the end of each year into perpetuity. Assume that the dividend is constant, and that the effective rate of interest is 5%

1. 18
2. 19
3. 20
4. 21
5. 22

Let K be the dividend received each year.

Then, $P = \frac{K}{i}$, where i is the interest rate.

$$\Rightarrow \frac{dP}{di} = -\frac{K}{i^2}$$

$$\Rightarrow D_{mac} = -(1+i)\frac{\frac{dP}{di}}{P} = \frac{1+i}{i} = \frac{1+0.05}{0.05} = 21$$

Exercise 2

John purchased three bonds to form a portfolio as follows

- Bond A has semi-annual coupons at 4%, a duration of 21.46 years, and was purchased for 980.
- Bond B is a 15-year bond with a duration of 12.35 years and was purchased for 1015.
- Bond C has a duration of 16.67 years and was purchased for 1000.

Calculate the Macaulay duration of the portfolio at the time of purchase.

1. 16.62 years
2. 16.67 years
3. 16.72 years
4. 16.77 years
5. 16.82 years

The present value of the portfolio is = 2995

$$D_{mac} = w_A D^A_{mac} + w_B D^B_{mac} + w_C D^C_{mac}$$

$$= \frac{980}{2995} \times 21.46 + \frac{1015}{2995} \times 12.35 + \frac{1000}{2995} \times 16.67$$

$$= 16.77$$

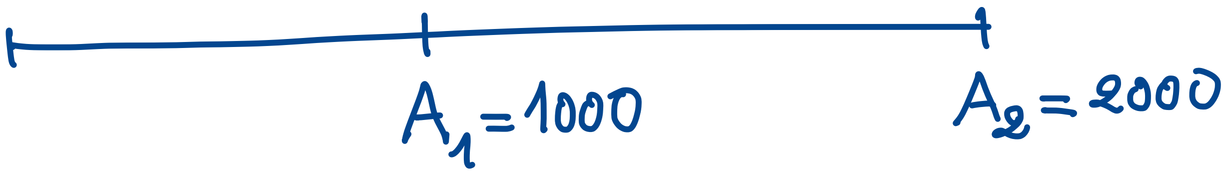
Exercise 3

A company must pay liabilities of 1000 and 2000 at the end of years 1 and 2, respectively. The only investments available to the company are the following two zero-coupon bonds

Maturity (years)	Effective annual yield	Par
1	10%	1000
2	12%	1000

Determine the cost to the company today to match its liabilities exactly.

1. 2007
2. 2259
3. 2503
4. 2756
5. 3001



The cost is $= A_1 v_1 + A_2 v_2^2$

$= 1000 \times 1.1^{-1} + 2000 \times 1.12^{-2}$

$= 2503.48$

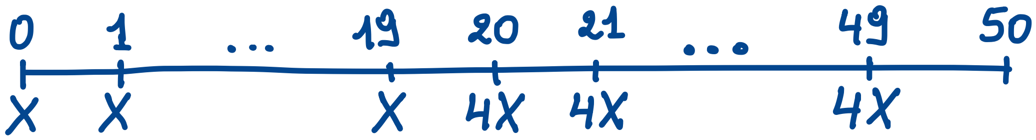
Exercise 4

An annuity provides the following payments:

- X at the beginning of each year for 20 years, starting today
- $4X$ at the beginning of ech year for 30 years, starting 20 years from today

Calculate the Macaulay duration of this annuity using an annual effective interest rate of 2%.

1. 27.32
2. 27.87
3. 28.30
4. 33.53
5. 35.41



$$D_{mac} = \frac{\sum_{t=0}^{19} tXv^t + \sum_{t=20}^{49} 4tXv^t}{\sum_{t=0}^{19} Xv^t + \sum_{t=20}^{49} 4Xv^t} = \frac{4 \cdot \sum_{t=1}^{49} tv^t - 3 \cdot \sum_{t=1}^{19} tv^t}{4\sum_{t=0}^{49} v^t - 3\sum_{t=0}^{19} v^t}$$

$$= \frac{4(Ia)_{\overline{49}|0.02} - 3(Ia)_{\overline{19}|0.02}}{4\ddot{a}_{\overline{50}|0.02} - 3\ddot{a}_{\overline{20}|0.02}}$$

$$= \frac{4 \cdot \frac{\ddot{a}_{\overline{49}|0.02} - 49 \cdot 1.02^{-49}}{0.02} - 3 \cdot \frac{\ddot{a}_{\overline{19}|0.02} - 19 \cdot 1.02^{-19}}{0.02}}{4\ddot{a}_{\overline{50}|0.02} - 3\ddot{a}_{\overline{20}|0.02}}$$

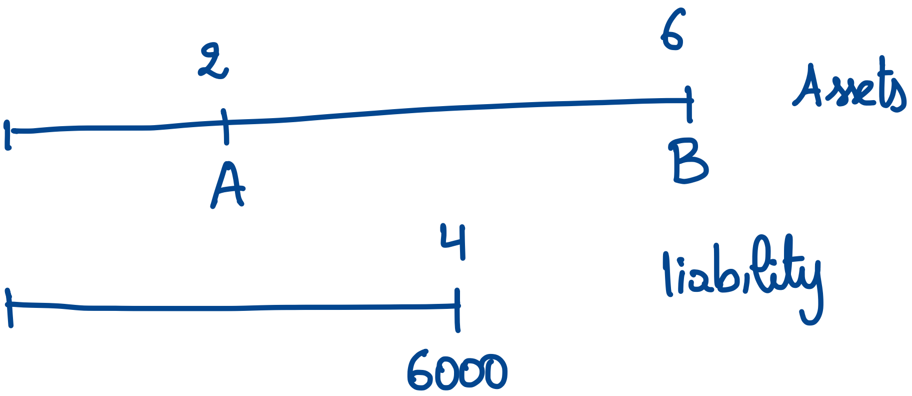
$$= \frac{2620.83 - 442.477}{128.21 - 50.035} = 27.865$$

Exercise 5

Aakash has a liability of 6,000 due in four years. This liability will be met with payments A in two years and B in six years. Aakash is employing a Redington immunization strategy using an annual effective interest rate of 5%.

Calculate $|A - B|$

1. 0
2. 146
3. 293
4. 586
5. 881



$$6000 \times 1.05^{-4} = A \times 1.05^{-2} + B \times 1.05^{-6} \quad (1)$$

$$4 \times 6000 \times 1.05^{-4} = 2 \times A \times 1.05^{-2} + 6 \times B \times 1.05^{-6} \quad (2)$$

$$(2) - 2 \times (1) \Rightarrow \quad 4B \times 1.05^{-6} = 9872.43 \Rightarrow B \approx 3307.5$$

$$(1) \Rightarrow \quad A = 1.05^{2}\left[6000 \times 1.05^{-4} - B \times 1.05^{-6}\right]$$

$$\approx 2721.09$$

$$\Rightarrow |A - B| \approx 586.41$$

Exercise 6

Yield rates to maturity for zero coupon bonds are currently quoted at 8.5% for one-year maturity, 9.5% for two-year maturity, and 10.5% for three-year maturity. Let i be the one-year forward rate for year two implied by current yields of these bonds.

Calculate i

1. 8.5%
2. 9.5%
3. 10.5%
4. 11.5%
5. 12.5%

$$i_{1,2} = \frac{(1+s_2)^2}{1+s_1} - 1$$

$$= \frac{1.095^2}{1.085} - 1$$

$$\simeq 10.51\%$$

Exercise 7

You are given the following information with respect to a bond:

- par value: 1000
- term to maturity: 3 years
- annual coupon rate: 6% payable annually

s_3 s_1 s_2

You are also given that the one, two, and three year annual spot interest rates are 7%, 8%, and 9% respectively. Calculate the value of the bond.

1. 906
2. 926
3. 930
4. 950
5. 1000

$Fr = 1000 \times 0.06 = 60$

	1	2	3
	60	60	1060

$$P = \frac{60}{1+s_1} + \frac{60}{(1+s_2)^2} + \frac{1060}{(1+s_3)^3}$$

$\simeq 926.03$

Exercise 8

A 20-year bond priced to have an annual effective yield of 10% has a Macaulay duration of 11. Immediately after the bond is priced, the market yeild rate increases by 0.25%. The bond's approximate percentage price change, using a first-order modified approximation, is X.

Calculate X.

1. -2.22%
2. -2.47%
3. -2.5%
4. -2.62%
5. -2.75%

$h = 0.0025$; $i_0 = 0.10$; $D_{mac} = 11$

$$P(i) \simeq P(i_0)\left[1 - h * \frac{D_{mac}}{1+i_0}\right]$$

$$\simeq 0.975\, P(i_0)$$

$$\Longrightarrow \frac{P(i)}{P(i_0)} \simeq 0.975$$

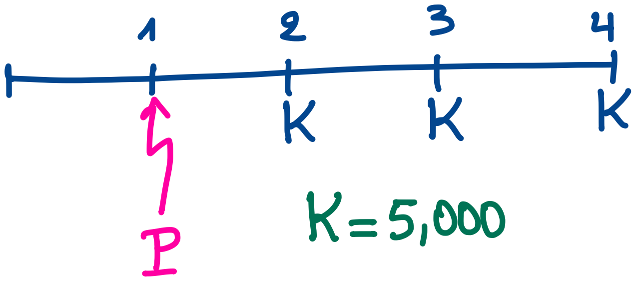
Exercise 9

You are given the following term structure of spot interest rates:

Term (in years)	Spot interest rate
1	5.00%
2	5.75%
3	6.25%
4	6.50%

A three year **annuity-immediate** will be issued **a year from now** with annual payments of 5000. Using the forward rates, calculate the present value of this annuity **a year from now**;

1. 13, 094
2. 13, 153
3. 13, 296
4. 13, 321
5. 13, 401



$K = 5{,}000$

$$P = \frac{K}{1+i_{1,2}} + \frac{K}{(1+i_{1,2})(1+i_{2,3})} + \frac{K}{(1+i_{1,2})(1+i_{2,3})(1+i_{3,4})}$$

$$= K\left[\frac{1+s_1}{(1+s_2)^2} + \frac{1+s_1}{(1+s_3)^3} + \frac{1+s_1}{(1+s_4)^4}\right]$$

$$\simeq 13{,}152.50$$

Exercise 10

A 30-year bond with a par value of 1000 and 12% coupons payable quarterly is selling at 850. Calculate the annual yield rate convertible quarterly.

1. 3.5%
2. 7.1%
3. 14.2%
4. 14.9%
5. 15.4%

$r = \frac{12\%}{4} = 3\%$; $n = 30 \times 4 = 120$; $F = 1000$

$P = 850$

Using the financial calculator, we get

$j^{(4)}/4 = 3.5392\%$

$\Rightarrow j^{(4)} = 14.1568\%$

Exercise 11

A ten-year 100 par value bond pays 8% coupons semiannually. The bond is priced at 118.20 to yield an annual nominal rate of 6% convertible semiannually.

Calculate the redemption value of the bond.

1. 97
2. 100
3. 103
4. 106
5. 109

$n = 10 \times 2 = 20$; $F = 100$; $r = \frac{8\%}{2} = 4\%$

$P = 118.20$; $j = \frac{6\%}{2} = 3\%$

$P = Cv^n + Fr\, a_{\overline{n}|j}$

$\Rightarrow C = (1+j)^n \left[P - Fr\, a_{\overline{n}|j} \right]$

$= 1.03^{20} \left[118.20 - 4 \times a_{\overline{20}|0.03} \right]$

$\simeq 106$

Exercise 12

You are given the following information with respect to a bond

- par value: 1000
- term to maturity: 3 years
- annual coupon rate: 6% payable annually

You are given that the one, two, and three year annual spot interest rates are 7%, 8%, and 9% respectively. The bond is sold at a price equal to its value.

Calculate the annual effective yield rate for the bond.

1. 8.1%
2. 8.3%
3. 8.5%
4. 8.7%
5. 8.9%

$$P = \frac{60}{1+s_1} + \frac{60}{(1+s_2)^2} + \frac{1060}{(1+s_3)^3} = 926.03$$

Using a financial calculator, we obtain

$j = 8.9179\ \%$

Exercise 13

A 40-year bond is purchased at a discount. The bond pays annual coupons. The amount for accumulation of discount in the 15th coupon is 194.82. The amount for accumulation of discount in the 20th coupon is 306.69.

Calculate the amount of discount in the purchase price of this bond

1. 13,635
2. 13,834
3. 16,098
4. 19,301
5. 21,135

(1) $BV_t = Cv^{n-t} + Fr(v + v^2 + \dots + v^{n-t})$

(2) $BV_{t-1} = Cv^{n-t+1} + Fr(v + \dots + v^{n-t} + v^{n-t+1})$

(1)-(2) $\Rightarrow BV_{t-1} - BV_t = Cv^{n-t} - Cv^{n-t+1} - Frv^{n-t+1}$

$= Cv^{n-t+1}\underbrace{(v^{-1} - 1)}_{j} - Frv^{n-t+1}$

$= (Cj - Fr)v^{n-t+1}$ = the amount of accumulation in the t-th year

$\begin{cases} (Cj - Fr)v^{26} = 194.82 & (3) \\ (Cj - Fr)v^{21} = 306.92 & (4) \end{cases}$

$\frac{(4)}{(3)} \Rightarrow (1+j)^5 = 1.575403$

$\Rightarrow j = 9.5162\%$

(3) $\Rightarrow Cj - Fr = 2070.48$

The amount of discount $= (BV_0 - BV_1) + \dots + (BV_{n-1} - BV_n)$

$= (Cj - Fr)\underbrace{(v^n + v^{n-1} + \dots + v)}_{a_{\overline{40|}j}} = 21,184$

Exercise 14

Toby purchased a 20-year par value bond with semiannual coupons at a nominal annual rate of 8% convertible semiannually at a price of 1722.25. The bond can be called at par value 1100 on any coupon date starting at the end if year 15.

What is the minimum yield that Toby could receive, expressed as a nominal annual rate of interest convertible semiannually?

1. 3.2%
2. 3.3%
3. 3.4%
4. 3.5%
5. 3.6%

$r = \frac{8\%}{2} = 4\%$; $n = 20 \times 2 = 40$; $P = 1722.25$

$F = C = 1100$;

As $P > F \Rightarrow r > j \Rightarrow$ bought at premium

The minimum maturity time is $= 15 \times 2 = 30$

Using a financial calculator, we get

$$\frac{j^{(2)}}{2} = 1.6082\% \Rightarrow j^{(2)} = 3.2164\%$$

Exercise 15

A 20-year of 1000 is repaid with payments at the end of each year. Each of the first ten payments equals 150% of the amount of interest due. Each of the last ten payments is X. The lender charges interest at an annual effective rate of 10%.

Calculate X

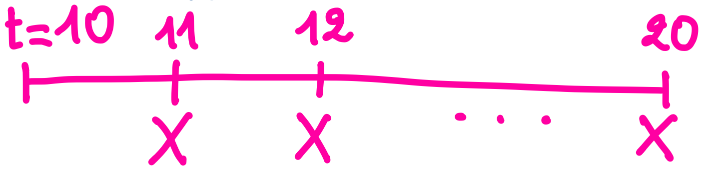
1. 32
2. 57
3. 70
4. 97
5. 117

For $t = 1, 2, \ldots, 10$, we have

$$I_t = OB_{t-1} \times i \Rightarrow PR_t = 0.5 I_t = 0.05\, OB_{t-1}$$

$$\Rightarrow OB_t = OB_{t-1} - 0.05\, OB_{t-1} = 0.95\, OB_{t-1}$$

$$\Rightarrow OB_{10} = (0.95)^{10} OB_0 = 598.737$$



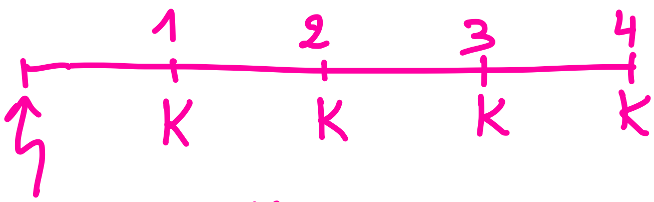
$$OB_{10} = X a_{\overline{10}|j} \Rightarrow X = \frac{OB_{10}}{a_{\overline{10}|j}} = 97.44$$

Exercise 16

Seth borrows X for four year at an annual effective interest rate of 8%, to be repaid with equal payments at the end of each year. The outstanding loan balance at the end of the ~~first~~ third year is 559.12.

Calculate the principal repaid in the first payment.

1. 444
2. 454
3. 464
4. 474
5. 484



$$OB_0 = X = K a_{\overline{4}|i}$$

$$i = 8\% ;\quad n = 4$$

$$OB_3 = Kv \Rightarrow K = (1+i)OB_3 = 603.85$$

$$\Rightarrow PR_1 = Kv^4 = 443.847$$

Exercise 17

A loan is repaid with level annual payments based on annual effective interest rate of 7%. The 8-th payment consists of 789 of interest and 211 of principal.

Calculate the amount of interest paid in the 18th payment.

1. 415
2. 444
3. 556
4. 585
5. 612

$$i = 7\% \; ; \; \begin{cases} I_8 = 789 = K(1 - v^{n-7}) & (1) \\ PR_8 = 211 = K v^{n-7} & (2) \end{cases}$$

$$\Rightarrow I_8 + PR_8 = K = 1000$$

$$(2) \Rightarrow (1+i)^{n-7} = \frac{K}{PR8} \Rightarrow n = 7 + \frac{\ln\left(K/PR8\right)}{\ln(1+i)}$$

$$= 30$$

$$\Rightarrow I_{18} = K(1 - v^{n-17}) = 585.035\$$$

Exercise 18

Bruce and Robbie each open up new bank accounts at time 0. Bruce deposits 100 into his bank account, and Robbie deposits 50 into his. Each account earns the same annual effective interest rate.

The amount of interest earned in Bruce's account during the 11th year is equal to X. The amount of interest earned in Robbie's account during the 17th year is equal to X.

Calculate X

1. 28.00
2. 31.30
3. 34.60
4. 36.70
5. 38.90

The amount of interest in Bruce's account during 11-th year

$$= 100 \times (1+i)^{11} - 100 \times (1+i)^{10}$$

$$= 100\, i\, (1+i)^{10} = X \quad (1)$$

The amount of interest in Robbie's account during 17-th year

$$= 50 \times (1+i)^{17} - 50 \times (1+i)^{16}$$

$$= 50\, i\, (1+i)^{16} = X \quad (2)$$

$$(1) / (2) \Rightarrow (1+i)^6 = 2 \Rightarrow i = \sqrt[6]{2} - 1 \simeq 12.25\%$$

$$\Rightarrow X = 100\, i\, (1+i)^{10} = 38.90\,\$$$

Exercise 19

At an annual interest rate of i, $i > 0\%$, the present value of a perpetuity paying 10 at the end of each 3-year period, with the first payment at the end of year 3, is 32.

At the same annual effective rate of i, the present value of a perpetuity paying 1 at the end of each 4-month period, with first payment at the end of 4 months, is X.

Calculate X.

1. 31.6
2. 32.6
3. 33.6
4. 34.6
5. 35.6

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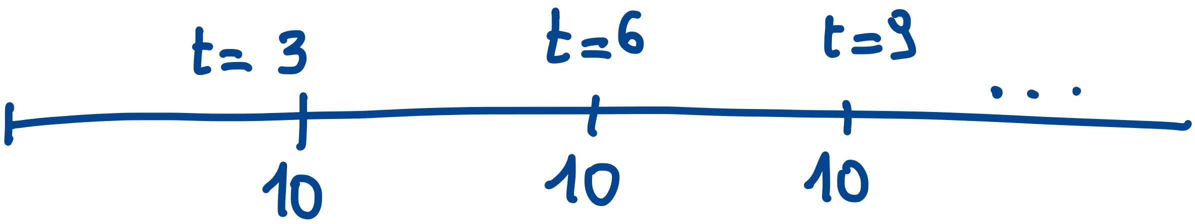

$$P = 32 = \frac{10}{\text{effective interest rate for three years}}$$

$$= \frac{10}{(1+i)^3 - 1} \Rightarrow i = 9.49\%$$

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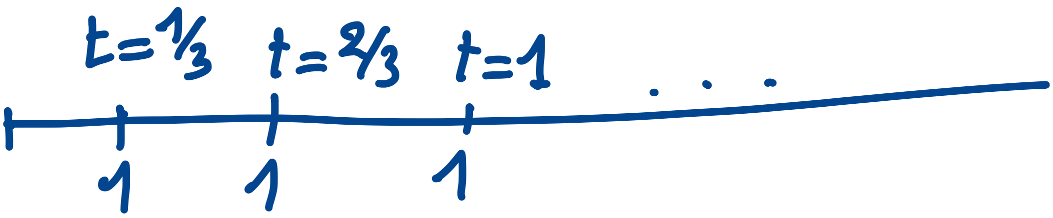

$$X = \frac{1}{(1+i)^{1/3} - 1} = 32.592$$

Exercise 20

You are given the following information about a loan of L that is to be repaid with a series of 16 annual payments:

- The first payment of 2000 is due one year from now.
- The next seven payments are each 3% larger than the preceding payment.
- From the 9th to the 16th payment, each payment will be 3% less than the preceding payment.
- The loan has an annual effective interest rate of 7%.

Calculate L.

1. 20,689
2. 20,716
3. 20,775
4. 21,147
5. 22,137

$r = 3\%;\ i = 7\%$

1	2	...	8	9	...	16
$K = 2000$	$K(1+r)$		$\underbrace{K(1+r)^7}_{K_8}$	$K_8(1-r)$	...	$K_8(1-r)^8$

$$L = K \times \frac{1-\left(\frac{1+r}{1+i}\right)^8}{i-r} + K_8(1-r) \times \frac{1-\left(\frac{1-r}{1+i}\right)^8}{i+r} \times v^8$$

$$= 2000 \times \frac{1-\left(\frac{1.03}{1.07}\right)^8}{0.04} + 2000 \times 1.03^7 \times 0.97 \times \frac{1-\left(\frac{0.97}{1.07}\right)^8}{0.1} \times 1.07^{-8}$$

$= 20{,}688.63$