Dept of Mathematics and Statistics
King Fahd University of Petroleum & Minerals
AS476: Survival Models for Actuaries
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Major 1 Exam Term 251 Sept. 22, 2025

Name_____________________________________……….____ID#:______

Instructions.

1. Please turn off your cell phones and place them under your chair. Any student caught with mobile phones on during the exam will be considered under the cheating rules of the University.
2. If you finish the test earlier and want to leave the room, please do so quietly so not to disturb others taking the test. No two person can leave the room at the same time. No extra time will be provided for the time missed outside the classroom.
3. Only materials provided by the instructor can be present on the table during the exam.
4. Do not spend too much time on any one question. If a question seems too difficult, leave it and go on.
5. Use the blank portions of each page for your work. Extra blank pages can be provided if necessary. If you use an extra page, indicate clearly what problem you are working on.
6. Only answers supported by work will be considered. Unsupported guesses will not be graded.
7. While every attempt is made to avoid defective questions, sometimes they do occur. In the rare event that you believe a question is defective, the instructor cannot give you any guidance beyond these instructions.
8. Mobile calculators, I-pad, or communicable devices are disallowed. Use regular scientific calculators or financial calculators only. Write important steps to arrive at the solution of the following problems.
9. Record your final answers to the multiple-choice questions on the OMR sheet. And write important steps to arrive at the solution of the last two problems.

The test is 110 minutes, GOOD LUCK, and you may begin now!

Question	**Total Mark**	**Mark Obtained**	**Comments**
1-5	**0.5*5**		
6a	**(1/3)*9**		
b-c	**0.5*2**		
7	**0.5**		
8-9	**1*2**		
10(i)	**2**		
ii	**3**		
11	**2**		
12	**2**		
13	**2**		
Bonus	**2**		
Total	**20**		

True or False (Circle T or F) for Questions 1-5:

T F 1. Mathematical models for survival analysis are frequently written in terms of a hazard function.

T F 2. One goal of a survival analysis is to compare survivor and/or hazard functions.

T F 3. Ordered failure times are censored data.

T F 4. Censored data are used in the analysis of survival data up to the time interval of censorship.

T F 5. A typical goal of a survival analysis involving several explanatory variables is to obtain an adjusted measure of effect.

6. Given the following survival time data (in weeks) for n = 15 subjects,
1, 1, 1+, 1+, 1+, 2, 2, 2, 2+, 2+, 3, 3, 3+, 4+, 5+
where + denotes censored data,

(a) complete the following table:

$t_{(f)}$	m_f	q_f	$R(t_{(f)})$
0	0	0	15 persons survive $\geq$ 0 weeks
1			
2			
3			

(b) compute the average survival time (T) and

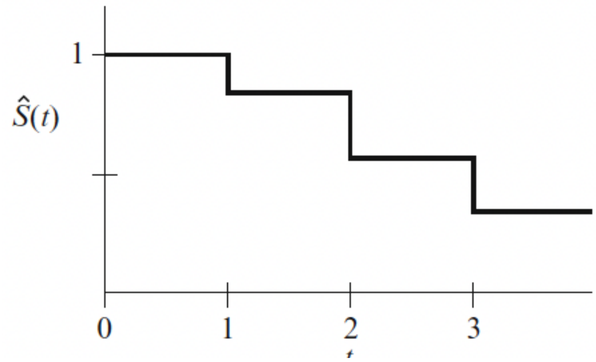
(c) the average hazard rate (h) using the raw data (ignoring + signs for T).

Solution: Practice Exercise 16, Chapter 1, KK textbook

$t_{(f)}$	m_f	q_f	$R(t_{(f)})$
0	0	0	15 persons survive $\geq$ 0 weeks
1	2	3	15 persons survive $\geq$ 1 weeks
2	3	2	10 persons survive $\geq$ 2 weeks
3	2	3	5 persons survive $\geq$ 3 weeks

$$\bar{T} = \frac{33}{15} = 2.2,; \quad \bar{h} = \frac{7}{33} = 0.22$$

7. Suppose that the estimated survivor curve for Q6 is given by the following graph:



What is the median survival time for this cohort?

Solution: Practice Exercise 17, Chapter 1, KK textbook

Median = 3 weeks.

The following summary is obtained for a mortality study data over a 5-year period. Answer the next two questions.

j	y_j	s_j	r_j
1	0.8	1	30
2	2.9	2	29
3	3.1	1	27
4	4.0	1	26
5	4.8	2	25

Note: s_j = frequency of death event and r_j = risk set.

8. Which of the following is the correct value of the Nelson-Aelen estimator for $H(2.9)$?

A) 0. 1103 B) 0.9028 C) 0.0333 D) 0.9000 E) 0. 1023

Solution: KPW ch 11 Example 11.4. Answer is E.

Table 11.7 Data for Example 11.4.

j	y_j	s_j	r_j	$S_{30}(x)$	$\hat{H}(x)$	$\hat{S}(x) = e^{-\hat{H}(x)}$
1	0.8	1	30	$\frac{29}{30} = 0.9667$	$\frac{1}{30} = 0.0333$	0.9672
2	2.9	2	29	$\frac{27}{30} = 0.9000$	$0.0333 + \frac{2}{29} = 0.1023$	0.9028
3	3.1	1	27	$\frac{26}{30} = 0.8667$	$0.1023 + \frac{1}{27} = 0.1393$	0.8700
4	4.0	1	26	$\frac{25}{30} = 0.8333$	$0.1393 + \frac{1}{26} = 0.1778$	0.8371
5	4.8	2	25	$\frac{23}{30} = 0.7667$	$0.1778 + \frac{2}{25} = 0.2578$	0.7727

9. Which of the following is the correct value of the Kaplan-Meier estimator for $S(2.9)$?

A) 0.1023 B) 0.1103 C) 0.9000 D) 0.9028 E) 0.9310

Solution Q8: KPW ch 12. Answer is C

Compare to example 12.2

$$\hat{S}(2.9) = \frac{30-1}{30}\left(\frac{29-2}{29}\right) = \frac{27}{30} = 0.9$$

10. A mortality study is based on observations during the period January 1, 2010, through December 31, 2012. Five policies were observed, with the following information recorded. For simplicity, a date of 3-1996 is interpreted as March 1, 1996, and all events are treated as occurring on the first day of the month of occurrence. Furthermore, all months are treated as being one-twelfth of a year in length. Summarize the information in a manner that is sufficient for estimating mortality probabilities.

1. Born 5-1977, purchased insurance policy on 8-2009, was an active policyholder on 1-2013.
2. Born 6-1977, purchased insurance policy on 7-2009, died 9-2011.
3. Born 8-1977, purchased insurance policy on 2-2011, surrendered policy on 2-2012.
4. Born 5-1977, purchased insurance policy on 6-2010, died 3-2011.
5. Born 7-1977, purchased insurance policy on 3-2010, surrendered policy on 5-2012.

A partial summary is given below.

policy holder	actual age at start	actual age at end	age group 32-33	33-34	34-35	35-36
1	a	b	c	12	12	d
2	32-7	34-3	5		3	
3	33-6		0	6		0
4	33-1	33-10	0	9	0	
5		34-10		12		0

Using the exact exposure method:

i) Which of the following **is correct**?

A) a = 32-8, b = 35-8, c = 4 months, d = 8 months, and under exact exposure $e_{32} = 13$

B) a = 32-5, b = 35-5, c = 7 months, d = 5 months, and under exact exposure $e_{32} = 16$

C) a = 32-6, b = 35-6, c = 6 months, d = 6 months, and under exact exposure $e_{32} = 15$

D) a = 32-8, b = 35-8, c = 4 months, d = 8 months, and under exact exposure $e_{32} = 14$

E) a = 32-9, b = 35-9, c = 3 months, d = 9 months, and under exact exposure $e_{32} = 12$

Solution: See KPW ch12 examples 12.14 and 12.15. Answer A. $e32$=c+5+0+0+4

ii) Estimate all mortality rates at integral ages, i.e., estimate e_{32}, e_{33}, e_{34}, e_{35}, $\hat{q}_{33}$, and $\hat{q}_{34}$, where e_j is a measure of exposure and $\hat{q}_j$ is the probability of death.

SOLUTION:

$$q_j = 1 - \exp(-d_j/e_j).$$

32-33: $e_{32} = 3 + 5 + 0 + 0 + 4 = 12.$

33-34: $e_{33} = 12 + 12 + 6 + 9 + 12 = 51.$

34-35: $e_{34} = 12 + 3 + 6 + 0 + 10 = 31.$

35-36: $e_{35} = 9 + 0 + 0 + 0 + 0 = 9.$

$$\hat{q}_{33} = 1 - \exp[-1/(51/12)] = 0.20966$$

$$\hat{q}_{34} = 1 - \exp[-1/(31/12)] = 0.32097.$$

11. For Data set A below, provide the empirical probability function at x=1 and the empirical distribution function at 1<= x < 2.

Table 11.1 Data Set A.

Number of accidents	Number of drivers
0	81,714
1	11,306
2	1,618
3	250
4	40
5 or more	7

Solution: Example 11.1, Chapter 11, KPW textbook

Emperical

probability function is

$$p_{94,935}(x) = \begin{cases} 81{,}714/94{,}935 = 0.860736, & x = 0, \\ 11{,}306/94{,}935 = 0.119092, & x = 1, \\ 1{,}618/94{,}935 = 0.017043, & x = 2, \\ 250/94{,}935 = 0.002633, & x = 3, \\ 40/94{,}935 = 0.000421, & x = 4, \\ 7/94{,}935 = 0.000074, & x = 5, \end{cases}$$

where the values add to 0.999999 due to rounding. The empirical distribution function is a step function with jumps at each data point.

$$F_{94,935}(x) = \begin{cases} 0/94{,}935 = 0.000000, & x < 0, \\ 81{,}714/94{,}935 = 0.860736, & 0 \le x < 1, \\ 93{,}020/94{,}935 = 0.979828, & 1 \le x < 2, \\ 94{,}638/94{,}935 = 0.996872, & 2 \le x < 3, \\ 94{,}888/94{,}935 = 0.999505, & 3 \le x < 4, \\ 94{,}928/94{,}935 = 0.999926, & 4 \le x < 5, \\ 94{,}935/94{,}935 = 1.000000, & x \ge 5. \end{cases}$$

12. All members of a study joined at birth; however, some may leave the study by means other than death. At the time of the third death, there was one death (i.e., $s_3 = 1$); at the time of the fourth death, there were two deaths; and at the time of the fifth death, there was one death. The following product-limit estimates were obtained:

$$S_n(y_3) = 0.72, S_n(y_4) = 0.60, \text{and} S_n(y_5) = 0.50.$$

Determine the number of censored observations between times y_4 and y_5. Assume no observations were censored at the death times.

SOLUTION

Step 1. Recall the product-limit form

At a death time y_j:

$$S_n(y_j) = S_n(y_{j-1})(1 - \frac{d_j}{r_j}),$$

where

- d_j= number of deaths at time y_j,
- r_j= number at risk just before y_j.

Censoring only reduces the risk set *between* death times.

Step 2. From given info

- At y_3: $d_3 = 1$, $S_n(y_3) = 0.72$.
- At y_4: $d_4 = 2$, $S_n(y_4) = 0.60$.
- At y_5: $d_5 = 1$, $S_n(y_5) = 0.50$.
- $S_n(y_2)$is not given, but we don't need it; we can work relatively.

Step 3. Ratio at each step

At y_4:

$$\frac{S_n(y_4)}{S_n(y_3)} = \frac{0.60}{0.72} = 0.8333.$$

So:

$$1 - \frac{d_4}{r_4} = 0.8333 \Rightarrow \frac{d_4}{r_4} = 0.1667.$$

Since $d_4 = 2$:

$$r_4 = \frac{2}{0.1667} = 12.$$

At y_5:

$$\frac{S_n(y_5)}{S_n(y_4)} = \frac{0.50}{0.60} = 0.8333.$$

So:

$$1 - \frac{d_5}{r_5} = 0.8333 \Rightarrow \frac{d_5}{r_5} = 0.1667.$$

Since $d_5 = 1$:

$$r_5 = \frac{1}{0.1667} = 6.$$

Step 4. Relating risk sets

We found:

- At y_4: $r_4 = 12$. After 2 deaths at y_4, survivors = 10.
- At y_5: $r_5 = 6$.

That means **between y_4 and y_5, 4 were censored** (since 10 survivors → only 6 remained at risk before y_5).

13. A mortality study has right censored data and no left truncated data. Uncensored observations occurred at ages 3, 5, 6, and 10. The risk sets at these ages were 50, 49, k, and 21, respectively, while the number of deaths observed at these ages were 1, 3, 5, and 7, respectively. The Nelson-Åalen estimate of the survival function at time 10 is 0.575. Determine k

    SOLUTION

Step 1. Nelson–Åalen cumulative hazard

$$\hat{H}(t) = \sum_{y_j \le t} \frac{d_j}{r_j},$$

where d_j= deaths at time y_j, r_j= risk set just before y_j.

The **Nelson–Åalen survival estimate**:

$$\hat{S}(t) = \exp\left(-\hat{H}(t)\right).$$

Step 2. Given data

- At age 3: $r = 50,\ d = 1$.
- At age 5: $r = 49,\ d = 3$.
- At age 6: $r = k,\ d = 5$.
- At age 10: $r = 21,\ d = 7$.

And $\hat{S}(10) = 0.575$.

Step 3. Compute hazard sum

$$\hat{H}(10) = \frac{1}{50} + \frac{3}{49} + \frac{5}{k} + \frac{7}{21}.$$

Step 4. Relating to survival

$$\hat{S}(10) = \exp\left(-\hat{H}(10)\right) = 0.575.$$

So

$$\hat{H}(10) = -\ln(0.575).$$

Compute: $-\ln(0.575) \approx 0.5539$.

Step 5. Substitute values

$$\frac{1}{50} + \frac{3}{49} + \frac{5}{k} + \frac{7}{21} = 0.5539.$$

Compute constants:

- $1/50 = 0.0200$.
- $3/49 \approx 0.0612$.
- $7/21 = 0.3333$.

Sum = 0.4145.

So:

$$0.4145 + \frac{5}{k} = 0.5539.$$
$$\frac{5}{k} = 0.1394.$$
$$k = \frac{5}{0.1394} \approx 35.9.$$

Answer: $k \approx 36$.

BONUS QUESTION (2 MARKS)

Let n be the number of lives observed from birth. None were censored and no two lives died at the same age. At the time of the ninth death, the Nelson-Åalen estimate of the cumulative hazard rate is 0.511, and at the time of the tenth death it is 0.588. Estimate the value of the survival function at the time of the third death.

SOLUTION

We are told this is **uncensored data, no ties**, so we can use the structure of the **Nelson–Åalen estimator** directly.

Step 1. Recall the Nelson–Åalen estimator

$$\widehat{H}(t) = \sum_{j: y_j \le t} \frac{1}{r_j},$$

since each death is 1 event and no censoring.
Here r_j= risk set just before the j-th death.

Since no censoring and $n$lives initially,

$$r_j = n - (j - 1).$$

So at death j, increment is $1/(n - j + 1)$.

Step 2. Use given hazard values

At the **9th death**:

$$\widehat{H}(y_9) = \sum_{j=1}^{9} \frac{1}{n - j + 1} = 0.511.$$

At the **10th death**:

$$\widehat{H}(y_{10}) = \widehat{H}(y_9) + \frac{1}{n - 9} = 0.588.$$

So:

$$\frac{1}{n - 9} = 0.588 - 0.511 = 0.077.$$
$$n - 9 = \frac{1}{0.077} \approx 12.987 \implies n \approx 22.$$

So initial cohort size $n = 22$.

Step 3. Estimate survival at the 3rd death

At the 3rd death:

$$\widehat{H}(y_3) = \frac{1}{22} + \frac{1}{21} + \frac{1}{20}.$$

Compute:

- $1/22 \approx 0.0455$,
- $1/21 \approx 0.0476$,
- $1/20 = 0.05$.

Sum $\widehat{H}(y_3) \approx 0.1431$.

Step 4. Nelson–Åalen survival

$$\hat{S}(y_3) = \exp\left(-\widehat{H}(y_3)\right) = e^{-0.1431} \approx 0.867.$$

Answer: The estimated survival function at the 3rd death is

$$\boxed{0.867 \text{ (approx)}}$$