King Fahd University of Petroleum and Minerals

Department of Mathematics

AS 491
Exam-2- — Term 222
Sunday, April 02, 2022

Allowed Time: 90 minutes

Instructor: **Dr. Boubaker Smii**

Name: ______________________________

ID #: ____________________

Section #: ___________ Serial Number: ___________

Instructions:

1. Write clearly and legibly. You may lose points for messy work.
2. **Show all your work**. No points for answers without justifications !

Question #	Grade	Maximum Points
1		12
2		10
3		09
4		09
5		12
6		08
Total:		60

Exercise 1:(12)
Consider the standard Brownian motion $\{B_t,\, t \geq 0\}$.

A)- Verify that $C_t = t\, B_{\frac{1}{t}},\ t > 0\,;\ C_0 = 0,$ is also a Brownian motion.

B)- Let $X_t = \mu\, t\, +\, \sigma\, B_t, \quad \sigma > 0,\ \mu \in \mathbb{R}$.
i)- Find the expectation and variance of X_t.

ii)- What kind of stochastic process is then X_t ?

iii)-What determines the quantity $\mu\, t$ in the process X_t ?

Exercise 2:(10)
Let $\{B_t,\, t \geq 0\}$ be the standard Brownian motion.

A)- Consider the stochastic process $X_t = B_t - t\,B_1,\ 0 \leq t \leq 1$.
a) What is called the process X_t?

b)Find $\mathbb{E}(X_t)$ and $\mathrm{Var}(X_t)$.

B)- Let $\{Y_t,\, t \geq 0\}$ be a stochastic process which satisfies: $Y_t = 1 + 0.2\,t + 0.5\,B_t$.
Find $\mathbf{P}[Y_{10} > 1 \mid Y_0 = 1]$.
(Express the result as $\Phi(x)$, where Φ is the standard normal distribution function and x a real number.)

Exercise 3:(09)
Consider the geometric Brownian motion given by

$$X_t = e^{\mu t + \sigma B_t}, \ t \geq 0, \ \sigma > 0, \ \mu \in \mathbb{R}. \tag{a}$$

A)-1- Give an application where we can use the geometric Brownian motion X_t.

2- Explain why the classical integration fails to integrate the Brownian sample paths.

B)- We assume that the price $X_t, t \geq 0$ of a risky asset (called stock) at time t is given by a geometric Brownian motion of the form:

$$X_t = X_0 \, e^{\mu t + \sigma B_t}, \quad (\mu = c - \frac{1}{2}\sigma^2), \quad t \geq 0, \ \sigma > 0, \ \mu \in \mathbb{R}. \tag{b}$$

1- What represent the quantity $c\,dt + \sigma\,dB_t$ during the period time $[t, t + dt]$?

2- Give a brief description of the following :

a- $\sigma\,dB_t$:

b- The constant c:

c- The constant $\sigma > 0$:

Exercise 4:(09)

Let $\{B_t,\, t \geq 0\}$ be the standard Brownian motion .

1. Find $\displaystyle\int_0^t B_s\, dB_s$.

2. Find $\displaystyle\int_0^t B_s^2\, dB_s$.

3. Find e^{B_t}.

Exercise 5: (12)

In the following $B(t)$, $B_1(t)$ and $B_2(t)$ are standard Brownian motion.

A- Let $X_t = e^{B_1(t)} \cos(B_2(t)),\ Y_t = e^{B_1(t)} \sin(B_2(t)),\ Z_t = e^{B_1(t)}$.

Write down dX_t, dY_t and dZ_t as $\alpha\, Y_t + \beta\, X_t + aZ_t,\ \gamma\, X_t + \eta\, Y_t + bZ_t$ and $\delta\, Z_t + cX_t + d\, Y_t$ respectively. (Here α, β, γ, η, δ, a, b, c, d) are coefficients to be determined !

B- The Black-Scholes-Merton model for growth with uncertain rate of return, is the value of §1 after time t, invested in a saving account. It is described by the following stochastic differential equation:

$$dX_t = \mu\, X_t\, dt + \sigma\, X_t\, dB_t,\ \mu, \sigma > 0 \tag{c}$$

1- Give the type of the SDE (c).

2- Verify that the solution of the SDE (c) is given by a Geometric Brownian motion.

Exercise 6: (08)
Let $\{B_t,\, t \geq 0\}$ be the standard Brownian motion .
A- Find the Itô process $X(t) = X_t,\, t \geq 0$ which verify the Stochastic differential equation:

$$dX_t = \frac{1}{2}\, X_t\, dt + X_t\, dB_t. \tag{d}$$

B- Verify that the process $X_t = \frac{B_t}{1+t}$ solves the Stochastic differential equation:

$$dX_t = -\frac{1}{1+t}\, X_t\, dt + \frac{1}{1+t}\, dB_t;\;\; X_0 = 0 \tag{e}$$