King Fahd University of Petroleum and Minerals

Department of Mathematics

AS 491
Final Exam- — Term 222
Monday, May 29, 2023
Allowed Time: 120 minutes

Instructor: **Dr. Boubaker Smii**

Name: ____________________________________

ID #: ______________________

Section #: ____________ Serial Number: ____________

Instructions:

1. Write clearly and legibly. You may lose points for messy work.
2. **Show all your work**. No points for answers without justifications !

Question #	Grade	Maximum Points
1		10
2		10
3		8
4		12
5		8
6		10
7		12
Total:		70

Exercise 1:(10)

A- An industrial psychologist has developed a test to identify people with potential to be good managers. His statistics indicate that 30 percent of the employees in the company have the potential to be good managers. When his test is given to a large number of employees similar to the ones in his company, 80 percent of the potentially good managers will pass the test while only 20 percent of the potentially poor managers will pass it. What is the probability that an employee who passes the test will be a good manager ?

B- Assume that the amount of fluid content contained in a can of soda has normal distribution with mean 12.2 ounces and standard deviation 0.1 ounces.

a)- Find the probability that a single can of soda contains less that 12 ounces.
(Express the result as $\Phi(x)$, where Φ is the standard normal distribution function and x a real number.)

b)- Suppose that 50 cans are measured, and that the 50 measurements are averaged. Assuming independence, find the probability that the average of these 50 measurements is less than 12.17.
(Express the result as $\Phi(c)$, where Φ is the standard normal distribution function and c a real number.)

Exercise 2:(10)

A continuous random variable X is said to have a Laplace distribution with parameter λ if its probability density function (pdf) is given by:

$$f(x) = A \exp(-\lambda \mid x \mid), \quad -\infty \leq x \leq \infty, \tag{a}$$

for some constant A.

1. Can the parameter λ be negative? Can λ be zero? Explain clearly your answer !

2. Compute the constant A in terms of λ . Sketch the pdf.

3. Compute the mean of X.

4. Compute the distribution function of X in terms of λ.

Exercise 3:(8)

Let S_t be the price of a stock at time t. Suppose that stock price is modelled as a geometric Brownian motion $S_t = S_0\, e^{\mu\, t + \sigma\, B_t}$, where B_t is a standard Brownian motion.

1- Suppose that the parameter values are $\mu = 0.055$ and $\sigma = 0.07$.
Given that $S_5 = 100$, find the probability that S_{10} is greater than 150. (you may express the result as $\Phi(x)$, where Φ is the standard Normal distribution function and x a real number.)

2- Now assume that μ and σ are fixed parameters and the initial value of the stock is $S_0 = 1$.
i)- Find the median of S_t and the expectation of S_t.
ii)- Given that $\mu = -\frac{1}{2}\sigma^2$. State, with justifications, whether or not the stock would be a good long term investment in this case.

Exercise 4: (12)
Let X_t, Y_t be Itô processes in $\mathbb{R}$.
1)- Verify that:

$$X_t\,Y_t = X_0\,Y_0 + \int_0^t Y_s\,dX_s + \int_0^t X_s\,dY_s + \int_0^t dX_s\,dY_s$$

2)- Let $F_t = \exp(-\,\alpha\,B_t + \frac{1}{2}\,\alpha^2\,t),\ \ \alpha\ \in\ \mathbb{R}$.
Find dF_t.

B- The mean reverting Ornstein-Uhlenbeck process is the solution X_t of the stochastic differential equation:

$$dX_t = (m - X_t)\,dt + \sigma\,dB_t, \tag{b}$$

where m, σ are real constants and $B_t\ \in\ \mathbb{R}$.
1- Find the solution X_t of the stochastic differential equation (b)

2-Find $\mathbb{E}(X_t)$.

Exercise 5: (8)
Let $\{B_t,\, t \geq 0\}$ be a standard Brownian motion.
Consider a model where the stock prices follow a geometric Brownian motion $S(t)$, given by the stochastic differential equation:

$$dS(t) = \mu\, S(t)\, dt + \sigma\, S(t)\, dB_t \tag{c}$$

1. Solve the stochastic differential equation (c).

2. Find the probability that at a certain time $t_1 > 0$ we will have negative prices.
(Express the result as $\Phi(x)$, where Φ is the standard Normal distribution function and x a given real number.)

<u>Exercise 6:</u> (10)

To describe the motion of a pendulum with small, random perturbations in its environment we consider the stochastic differential equation :

$$Y_t'' + \Big(1 + \epsilon W_t\Big) Y_t = 0; \qquad Y_0,\, Y_0' \ \textit{given}, \tag{d}$$

where W_t is a one-dimensional white noise, ϵ a positive constant.

Verify that the stochastic differential equation (d) can be written in the following form:

$$dX_t = K\, X_t\, dt - \epsilon\, L\, X_t\, dB_t, \tag{e}$$

where X_t, K, L are suitable matrices and B_t a Brownian motion.

Exercise 7: (12)
Let $\{B_t,\, t \geq 0\}$ be a standard Brownian motion and $X_t = X_0 + c\int_0^t X_s\, ds + \sigma\int_0^t X_s\, dB_s$, $A_t = A_0\, e^{rt}$.
A- Assume that the value of a **portfolio** (wealth) at time t is given by:

$$V_t = a_t\, X_t + b_t\, A_t = u(T-t, X_t)\ ,\ t \in [0,T]$$

for some smooth deterministic function $u(t,x)$.
(You want to hold certain amounts of shares: a_t in stock and b_t in bond).
Describe the following situation:

- $a_t < 0$:

- $b_t < 0$:

B- For $s < t$ we can buy (or sell) shares of the stock at price $X(s)$ and then sell (or buy) these shares at time t for the price $X(t)$.
Suppose that the option gives us the right to buy one share of the stock at time t for a price K.

1- Give the expression of the worth of option at time t.

2- Write down the expression of V_t in terms of u_1, u_2, u_{22} and X_t.

C- Let Φ be the standard normal distribution and g, h given functions.
The **Black-Scholes** option pricing formula is given by

$$\begin{aligned} V_0 &= u(T, X_0) \\ &= X_0\, \Phi(g(T,X_0)) - Ke^{-rT}\, \Phi(h(T,X_0)), \end{aligned}$$

which is a rational price at time $t = 0$ for an European option with exercise price K.

1- Find the value of your self-financing portfolio at time $t \in [0,T]$.

2- Is the value of the self-financing portfolio at time $t = 0$ depend on c ? σ ?