Math 514 Comprehensive 1

Name:.....ID#:.....

Q.1 (10 points) Evaluate the integral $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2(x^2+4)}\, dx$ using residue theory.

Q.2 (10 points) Use Laplace transform to evaluate the integral

$$\int_0^{\infty} \frac{\sin(xt)}{x(x^2+4)} dx$$

Q.3 (15 points) Use Laplace transform to solve the heat equation

$$ku_{xx} = u_t,\ 0 < x < L,\ t > 0$$

under the following conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0$$

and

$$u(x,0) = x \quad 0 < x < L$$

Apply residue theory and write final answer as a series.

Q.4 (10 points) Solve the integral equation using Fourier transform

$$\int_{-\infty}^{\infty} e^{-4t^2} f(x-t) dt = e^{-3x^2}.$$

Q.5 (10 points) Use Fourier transform to solve the heat equation

$$ku_{xx} = u_t,\ -\infty < x < \infty,\ t > 0$$

under the following conditions $u(x,0) = f(x)$ where $f(x) = \begin{cases} u_o & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$.

Q.6 (10 points) Solve using Mellin transform

$$x^2 u_{xx} + x u_x + y_{yy} = 0, \quad 0 \le x < \infty, \quad 0 < y < 1$$

under the conditions $u(x,0) = 0$ and $u(x,1) = \begin{cases} A & 0 < x < 1 \\ 0 & x > 1 \end{cases}$.

Q.7 (10 points) Show the Hankel transform

$$\mathcal{H}_o\{e^{-ar}\} = \frac{a}{(a^2+\alpha^2)^{3/2}}$$

Q.8 (10 points) Solve the axisymmetric bihormonic equation using Hankel transform

$$\begin{aligned} \nabla^4 u(r,z) &= 0, \quad 0 \le r < \infty, \quad z > 0 \\ u(r,0) &= f(r), \quad 0 \le r < \infty, \\ \frac{\partial u}{\partial z} &= 0 \quad on \quad z = 0, \quad 0 \le r < \infty \\ u(r,z) &\to 0 \quad as \quad z \to \infty. \end{aligned}$$

Q.9 (15 points) Solve the integral equation using Wiener-Hopf technique

$$\int_0^\infty e^{-|x-\xi|} u(\xi) d\xi = -\frac{1}{4} u(x) + 1, \quad 0 < x < \infty.$$

$u(x)$ is bounded as $x \to \infty$.