King Fahd University of Petroleum and Minerals
Department of Mathematics
Qualifying Exam-Real Analysis - Term 221
Time:180 Minutes

Notation: $\mathbb{R}$ = the real numbers, $\mathbb{N}$ = the natural numbers, m = Lebesgue measure, m^* = Lebesgue outer measure.

Instructions: Solving (Completely and correctly) six problems will result in a passing grade.

Justify your answers thoroughly. For any theorem that you wish to cite, you should give its name and its statement.

Exercise 1

Let C be a Lebesgue measurable set satisfying $m^*(C\Delta D) = 0$ for a given set D. Here $C\Delta D = (C \cap D^c) \cup (D \cap C^c)$. Show that D is measurable.

Exercise 2

Show that if f is a non-negative measurable function on E, then $f = 0$ a.e. on E is equivalent to $\int_E f = 0$.

Exercise 3

We define $f : [0,1] \longrightarrow \mathbb{R}$ by $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$

Find $m\{x : f(x) \geq 0\}$.

Exercise 4

Evaluate with proof

$$\lim_{n\to\infty} \int_0^n \frac{\cos\left(\frac{x}{n}\right)}{1+x} dx.$$

Exercise 5

Suppose that f is finite on $[a, b]$ and is of bounded variation on every interval $[a + \varepsilon, b]$, for $\varepsilon > 0$, with $TV(f_{[a+\varepsilon,b]}) \leq M < +\infty$.

1. Show that $TV(f_{[a,b]}) < +\infty$.
2. Is $TV(f_{[a,b]}) \leq M$? If not what additional assumptions will make it so?

Exercise 6

Let f and each f_n be integrable on E. If $\lim_{n\to\infty}\int_E |f_n - f| = 0$, show that $(f_n) \longrightarrow f$ in measure on E.

Exercise 7

Let $\{f_n\}_{n=1}^{\infty}$ be measurable on $[0,1]$. Show that if $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable on $[0,1]$, then there exists a constant $C < \infty$ such that $\int_{[0,1]} |f_n| \leq C$, for all $n \geq 1$.

Exercise 8

Prove that if $f \in L^1(\mathbb{R})$, then $\lim_{n\to\infty} \int_{|x|\geq n} |f(x)| = 0$. Demonstrate that it is not necessarily the case that $f(x) \to 0$ as $|x| \to \infty$.