PhD Comprehensive Exam

Duration: 180 minutes

ID:	
NAME:	

- Show your work.
- There are empty pages attached to this exam booklet.

Problem	Score
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Problem 1

Prove that the union of a finite collection of measurable sets is measurable.

Problem 2

Prove that if a σ-algebra of subsets of $\mathbb{R}$ contains intervals of the form (a, ∞), then it contains all intervals.

Problem 3

A function f is said to be **Borel measurable** provided its domain E is a Borel set and for each α, the set $\{x \in E \mid f(x) > \alpha\}$ is a Borel set. Show that

(a) every Borel measurable function is Lebesgue measurable,

(b) if f is Borel measurable and B is a Borel set, then $f^{-1}(B)$ is a Borel set.

Problem 4

Prove that

$$\lim_{n\to\infty} \int_0^1 (n+1)x^n \sin^{-1} x \ \ dx = \frac{\pi}{2}.$$

Problem 5

Let $f : [a, b] \longrightarrow [m, M]$ be an absolutely continuous function and $g : [m, M] \longrightarrow \mathbb{R}$ be Lipchitz. Show that $h = g \circ f$ is absolutely continuous on $[a, b]$.

Problem 6

(a) Let $1 \leq p < q < r < \infty$. Show that $L^p(\mathbb{R}) \cap L^r(\mathbb{R}) \subseteq L^q(\mathbb{R})$.

(b) Prove or give a counterexample for the following:

Suppose that $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ is Lebesgue measurable for each n such that $\|f_n\|_3 \to 0$ and $\|f_n\|_5 \to 0$ as $n \to \infty$, then we have $\|f_n\|_4 \to 0$.

Problem 7

Let $\{f_n\}_{n=1}^{\infty}$ be measurable on $[0,1]$. Show that if $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable on $[0,1]$, then there exists a constant $C < \infty$ such that $\int_{[0,1]} |f_n| \leq C$, for all $n \geq 1$.

Problem 8

Evaluate

$$\lim_{n\to\infty}\int_0^n \frac{\cos\left(\frac{x}{n}\right)}{1+x}dx.$$