PhD Comprehensive Exam

Duration: 180 minutes

ID:	
NAME:	

- Justify your answers thoroughly. For any theorem that you wish to cite, you should give its name and its statement.
- Instructions: Solving (Completely and correctly) six problems will result in a passing grade.

Problem	Score
1	
2	
3	
4	
5	
7	
6	
8	
Total	/100

Problem 1

Let $f : [0, 1] \longrightarrow [0, 1]$ be measurable. Show that

$$\lim_{n\to\infty} \int_0^1 \frac{1+n}{1+nf(x)} dx = \int_0^1 \frac{1}{f(x)} dx.$$

Problem 2

Let f and g be nonnegative functions in $L^1(\mathbf{R})$ and measurable on the interval $[0,1]$. In addition, suppose that

$$f(x)g(x) \geq 1 \text{ for all } x \in [0,1].$$

Show that

$$\left(\int_{[0,1]} f(x) \right) \left(\int_{[0,1]} g(x) \right) \geq 1.$$

Problem 3

Let f be a nonnegative measurbale function on $[0,1]$ such that for all integers $n \geq 1$, we have $\int_{[0,1]} f^n \leq \frac{2^n}{n^2}$. Show that $f(x) < 2$ for a.e. $x \in [0,1]$.

Problem 4

Let μ and ν be measures on the measurable space $(X, \mathcal{M})$. For $E \in \mathcal{M}$, define $v(E) = \max\{\mu(E), \nu(E)\}$. Is v a measure on $(X, \mathcal{M})$?

Problem 5

Let f and g be nonnegative measurable functions on X for which $g \leq f$ a.e. on X. Show that $f = g$ a.e. on X if and only if $\int_X g d\mu = \int_X f d\mu$.

Problem 6

Let $\{f_n\}_n$ be a sequence of functions in $L^2([0,1])$ satisfying $\|f_n\|_2 \leq M$ for all $n \geq 1$. In addition, suppose that $\{f_n\}_n \to f$ almost everywhere on $[0,1]$. Show that $f \in L^2([0,1])$ with $\|f\|_2 \leq M$

Problem 7

Suppose that $\{f_n\}$ is bounded in $L^1[0,1]$. Is $\{f_n\}$ uniformly integrable over $[0,1]$?

Problem 8

Let f and each f_n be integrable on E, for any $n \geq 1$. If $\lim_{n\to\infty} \int_E |f_n - f| = 0$, show that $(f_n) \longrightarrow f$ in measure on E.