PhD Comprehensive Exam

Duration: 180 minutes

ID:	
NAME:	

- Justify your answers thoroughly. For any theorem that you wish to cite, you should give its name and its statement.

Problem	Score
1	
2	
3	
4	
5	
6	
7	
8	
Total	/100

Problem 1

Let E have finite outer measure. Show that E is measurable if and only if for each open, bounded interval (a,b), $b-a = m^*((a,b)\cap E) + m^*((a,b)\sim E)$.

Problem 2

(a) For the function f and the set F in the statement of Lusin's Theorem, show that the restriction of f to F is a continuous function.

(b) Prove the extension of Lusin's Theorem to the case that f is not necessarily real-valued, but may be finite a.e.

Problem 3

Let $\{f_n\}$ be a sequence of integrable functions on E for which $f_n \to f$ a.e. on E and f is integrable over E. Show that $\int_E |f - f_n| \to 0$ if and only if $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$. (Hint: Use the General Lebesgue Dominated Convergence Theorem.)

Problem 4

Consider the sequence of functions

$$f_n(x) = \left(1 + \frac{x}{n}\right)^n.$$

(a) For all $n \geq 1$ and $x > -n$, show that f_n is monotone increasing and that

$$f_n(x) \leq e^x.$$

(b) Evaluate

$$\lim_{n\to\infty} \int_0^n f_n(x) e^{-5x} dx.$$

Problem 5

If f is continuous on $[a, b]$ and f' exists and is bounded on (a, b), then show that f is absolutely continuous on $[a, b]$.

Problem 6

Assume E has finite measure and $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence of measurable functions that converge pointwise a.e. on E to f. For $1 \leq p < \infty$, show that $\{f_n\} \to f$ in $L^p(E)$ if there is $\alpha > 0$ such that $\{f_n\}$ belongs to and is bounded as a subset of $L^{p+\alpha}(E)$.

Problem 7

Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{h_n\}$ be a sequence of nonnegative integrable functions on X. Suppose that $\{h_n(x)\} \to 0$ for almost all $x \in X$. Show that

$$\lim_{n\to\infty} \int_X h_n \, d\mu = 0$$

if and only if $\{h_n\}$ is uniformly integrable and tight over X.

Problem 8

Let $\mathbb{N}$ be the set of natural numbers, and let $\mathcal{M} = 2^{\mathbb{N}}$ and c be the counting measure by setting $c(E)$ equal to the number of points in E if E is finite and ∞ if E is infinite. Let $(X, \mathcal{A}, \mu) = (Y, \mathcal{B}, \nu) = (\mathbb{N}, \mathcal{M}, c)$. Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 2 - 2^{-x} & \text{if } \ x = y, \\ -2 + 2^{-x} & \text{if } \ x = y + 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that f is measurable with respect to the product measure $c \times c$.

(b) Show that

$$\int_{\mathbb{N}} \left[\int_{\mathbb{N}} f(m,n) dc(m) \right] dc(n) \neq \int_{\mathbb{N}} \left[\int_{\mathbb{N}} f(m,n) dc(n) \right] dc(m).$$

(c) Is this a contradiction either of Fubini's theorem or Tonelli's theorem ?