PhD Comprehensive Exam

Duration: 180 minutes

ID:	
NAME:	

- Justify your answers thoroughly. For any theorem that you wish to cite, you should give its name and its statement.

Problem	Score
1	
2	
3	
4	
5	
7	
6	
8	
Total	/100

Problem 1

Show that a set E is measurable if and only if for every $\varepsilon > 0$, there exists a closed set F and open set $\mathcal{O}$ for which $F \subseteq E \subseteq \mathcal{O}$ and $m^*(\mathcal{O} \sim F) \leq \varepsilon$.

Problem 2

Assume that E is a set of finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise on E a.e. to a real-valued function f that is finite a.e. Show that the conclusion of Egoroff's Theorem still holds.

Problem 3

Consider two Lebesgue integrable functions f and g over $\mathbb{R}$ and two sequences of Lebesgue integrable functions $\{f_n\}$ and $\{g_n\}$ over $\mathbb{R}$. Assume that

(i) $\{f_n\}$ converges to f pointwise a.e. on $\mathbb{R}$,

(ii) $\{g_n\}$ converges to g pointwise a.e. on $\mathbb{R}$,

(iii) $|f_n| \leq g_n$ a.e. on $\mathbb{R}$ and

(iv) $\int_{\mathbb{R}} g_n$ converges to $\int_{\mathbb{R}} g$.

Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f$$

Problem 4

Consider the sequence of functions

$$f_n(x) = \left(1 + \frac{x}{n}\right)^n.$$

(a) For all $n \geq 1$ and $x > -n$, show that f_n is monotone increasing and that

$$f_n(x) \leq e^x.$$

(b) Evaluate

$$\lim_{n\to\infty} \int_0^n f_n(x) e^{-3x} dx.$$

Problem 5

Let the function f be absolutely continuous on $[a, b]$. Show that f is Lipschitz on $[a, b]$ if and only if there exists $M > 0$ such that $f'(x) \leq M$ a.e. on $[a, b]$.

Problem 6

Let E be a measurable set and $1 \leq p < \infty$ and $f_n \to f$ in $L^p(E)$.

(a) If the Lebesgue measure, m, of E is finite that is $m(E) < \infty$, show that $f_n \to f$ in $L^s(E)$ for some $1 \leq s < p$.

(b) If $f_n \to f$ a.e on E and there exists a real number M such that $|f_n| \leq M$ a.e. for all n, show that $f_n \to f$ in $L^r(E)$ for some r such that $1 \leq p < r < \infty$.

Problem 7

Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{h_n\}$ be a sequence of nonnegative integrable functions on X. Suppose that $\{h_n(x)\} \to 0$ for almost all $x \in X$. Show that

$$\lim_{n\to\infty} \int_X h_n \, d\mu = 0$$

if and only if $\{h_n\}$ is uniformly integrable and tight over X.

Problem 8

Let $X = Y$ be the interval $[0, 1]$ with $\mathcal{A} = \mathcal{B}$ the class of Borel sets. Let $\mu = \nu$ be the Lebesgue measure. Consider the function f on $X \times Y$ defined as

$$f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}, \quad (x, y) \in X \times Y.$$

(a) Show that

$$\int_X \int_Y f d\mu d\nu \neq \int_Y \int_X f d\nu d\mu.$$

(b) Does part (a) contradict Fubini's theorem? why?