King Fahd University of Petroleum and Minerals
Department of Mathematics

Comprehensive Exam- Math 533
Complex Analysis
1 September 2021

Duration: 150 minutes

Name: ______________________________

This exam contains 8 pages (including this cover page) and 7 questions. Total of points is 100.

Distribution of Marks

Question	Points	Score
1	15	
2	20	
3	10	
4	10	
5	15	
6	15	
7	15	
Total:	100	

1. (a) (15 points) Find all entire functions f such that

$$|f(z)| \leq |z|^{2/3}, \text{ for all } z \in \mathbb{C}.$$

2. (a) (5 points) Let f be an entire function such that $|\mathrm{Im}\, f(z)| < \pi$ for all $z \in \mathbb{C}$. Show that f is constant.

   (b) (5 points) Show that there is no nonconstant entire function g such that $g(\mathbb{C}) \subset \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$.

   (c) (10 points)

      (i) Show that $T(z) = \dfrac{z-1}{z+1}$ maps $\mathbb{C} \setminus [-1, 1]$ onto $\mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$.

      (ii) Show that if h is an entire function such that $h(\mathbb{C}) \subset \mathbb{C} \setminus [-1, 1]$, then h is a constant.

3. (10 points) Describe all entire functions f such that

$$|f(z)| \leq |\sin(z)| \text{ for all } z \in \mathbb{C}.$$

4. Suppose that $f : \Delta \to \Delta$ is analytic on the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and continuous on $|z| = 1$ such that $f(0) = f(a) = 0$, where $a \in \Delta$, and $a \neq 0$.

   (a) (5 points) Show that $|f'(0)| \leq |a|$. (Hint: Consider $z\varphi_a(z)$, where $\varphi_a(z) = \dfrac{z-a}{1-\overline{a}z}$)

   (b) (5 points) Find f, if $|f'(0)| = |a|$.

5. (a) (10 points) Let $\mathscr{C}$ be a simply closed, positively oriented contour and f be an analytic function inside and on $\mathscr{C}$. Assume that f does not vanish on $\mathscr{C}$. Let $\{a_1, \ldots, a_N\}$ be the zeros of f inside $\mathscr{C}$ (counted with multiplicities). Show that

$$\frac{1}{2\pi i}\int_{\mathscr{C}} \frac{z f'(z)}{f(z)}\,dz = a_1 + \ldots + a_N.$$

(b) (5 points) Application: Let $n \in \mathbb{N}$, $n \geq 2$. Find

$$\oint\limits_{|z|=2} \frac{z^n}{z^n - 1}\,dz$$

6. Let $n \in \mathbb{N}$, $n \geq 1$ and $\alpha \in (-1, 1)$, with $\alpha \neq 0$.

   (a) (10 points) Compute $\displaystyle\int_0^{2\pi} \frac{e^{in\theta}}{(e^{i\theta}-\alpha)(e^{-i\theta}-\alpha)}\,d\theta$

   (b) (5 points) Deduce $\displaystyle I = \int_0^{2\pi} \frac{\cos n\theta}{1-2\alpha\cos\theta+\alpha^2}\,d\theta.$

7. (15 points) Let $f(z) = z^6 - 2z + 4$. Show that all the zeros of f lie on the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$.