King Fahd University of Petroleum and Minerals
Department of Mathematics
MATH533 - Complex Variables I
Comprehensive Exam – Term 212

Justify your answers thoroughly. For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Exercise 1

Compute

$$\int_0^{2\pi} \frac{\cos\theta}{\cos\theta - i}\, d\theta$$

Exercise 2

Let f and g be polynomials with $\deg(g) > \deg(f) + 1$.

(a) Show that $\lim_{R\to\infty} \int_{|z|=R} \frac{f(z)}{g(z)}\, dz = 0$.

(b) Use (a) to show that the sum of the residues of $\frac{f}{g}$ at all its poles is zero.

Exercise 3

Let f and g be two entire functions such that

$$|f(z)| \leq |g(z)| \text{ for all } z \in \mathbb{C}.$$

Show that $f = cg$, for some constant $c \in \mathbb{C}$ with $|c| \leq 1$.

Exercise 4

Let $\mathbb{D}$ be the unit disc and $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im(z)} > 0\}$.

(a) Show that the function $\varphi : \mathbb{H} \to \mathbb{D}$ defined by $\varphi(z) = \dfrac{z-i}{z+i}$ is an analytic bijection with an analytic inverse.

(b) Let $f : \mathbb{D} \setminus \{0\} \to \mathbb{H}$ be an analytic function. Study the nature of its singularity at zero.

Exercise 5

Let f be an analytic function on a nonempty open connected set $\Omega \subset \mathbb{C}$. Let $a \in \Omega$ be a local minimum of $|f|$.

(a) Prove that either $f(a) = 0$ or f is constant on Ω.

(b) Prove or disprove that there exists an analytic function f on the unit disc $\mathbb{D}$ such that $|f(z)|^2 = |z|^2 + 1$ for all $z \in \mathbb{D}$.

Exercise 6

Let (f_n) be a sequence of analytic functions inside and on $|z| = 1$. Suppose that f_n converges uniformly to f inside and on $|z| = 1$.

Show if f has no zeros on $|z| = 1$, then the number of zeros of f inside $|z| = 1$ is equal to the number of zeros of f_n inside $|z| = 1$ for sufficiently large n.

Exercise 7

Let $\Omega \subset \mathbb{C}$ be a *bounded domain* and let

$$f : \Omega \to \Omega$$

be an analytic function. Suppose that $f(z_0) = z_0$ for a point z_0 in Ω. Let

$$f_n := \underbrace{f \circ f \circ \cdots \circ f}_{n\text{-times}}$$

(a) Prove by induction that $(f_n)'(z_0) = (f'(z_0))^n$, for all $n \geq 1$.

(b) Prove that $|(f_n)'(z_0)| \leq C$ for all $n \geq 1$, for some constant C.

(c) Deduce that $|f'(z_0)| \leq 1$.

(d) In addition, assume that f is an automorphism of Ω. What is the value of $|f'(z_0)|$?