King Fahd University of Petroleum and Minerals
Department of Mathematics
MATH533 - Complex Variables
Comprehensive Exam – Term 231

Justify your answers thoroughly. For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

1. (20 points)

   Let $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the upper half plane. Prove the following

   (a) If $\alpha \notin H$ and $\beta \in H$, then $\dfrac{1}{\alpha - \beta} \in H$.

   (b) For any $\xi_1, \ldots \xi_k \in H$, $\xi_1 + \ldots + \xi_k \in H$, in particular, $\xi_1 + \ldots + \xi_k \neq 0$.

   (c) Let P be a polynomial with zeros $z_1, \ldots, z_k$ (possibly repeated)

   $$\frac{P'(z)}{P(z)} = \frac{1}{z - z_1} + \ldots + \frac{1}{z - z_k}.$$

   (d) Using the previous questions, prove the following:
   " Let P be a polynomial. Suppose all the zeros of P lie in H. Then all the zeros of P' also lie in H".

2. (15 points)

   Let f be an analytic function on $\Omega = \{z \in \mathbb{C} : |z| < 4\}$. Suppose that $|f(z)| < 1$ on Ω. Let

$$g(z) = f(z) + z - 2.$$

   (a) Prove that all the zeros of g lie in the disc $D = \{z \in \mathbb{C} : |z - 2| < 1\}$.

   (b) Using Rouché theorem, prove that g has only one zero inside Ω.

   (c) What is g if $\Omega = \mathbb{C}$?

3. (10 points) Evaluate

$$\int_0^{2\pi} \frac{\cos^2\theta}{5+3\sin\theta}d\theta.$$

4. (10 points) Let Ω be a bounded domain in the complex plane. Suppose that f is continuous on $\overline{\Omega}$ and analytic on Ω. Assume $|f(z)| = 1$ for all $z \in \partial\Omega$, the boundary of Ω.
   Show that f is a constant function or f has a zero on Ω.

5. (20 points) Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that $\lim_{|z|\to\infty} |f(z)| = +\infty$.

(a) Prove that f has a finite number of zeros in $\mathbb{C}$.

(b) Prove that there exists a polynomial P such that $f = \dfrac{P}{g}$, where g is holomorphic in $\mathbb{C}$ and $g(z) \neq 0$, for all $z \in \mathbb{C}$.

(c) Prove that there exists $R > 0$ such that $|g(z)| \leq |P(z)|$, for all $|z| \geq R$ and that g is a polynomial.

(d) Deduce that there exists a constant c such that $f = cP$.

6. (15 points) Let f be an analytic function on $\mathbb{C}$.

(a) Prove that for any $\alpha, \beta \in \mathbb{C}$, with $\alpha \neq \beta$, we have for $R > \max(|\alpha|, |\beta|)$

$$\frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{(z-\alpha)(z-\beta)} dz = \frac{f(\alpha) - f(\beta)}{\alpha - \beta}$$

(b) Show if f is bounded, then

$$\lim_{R \to \infty} \int_{|z|=R} \frac{f(z)}{(z-\alpha)(z-\beta)} dz = 0.$$

(c) Using ONLY (a) and (b), show that if f is analytic and bounded on $\mathbb{C}$, then f is constant. (No credit for other methods).

7. (10 points) Let

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

be an analytic function and

$$F : (x, y) \mapsto (u(x, y), v(v, y)).$$

(a) Show that

$$\det J_F(x, y) = |f'(z)|^2,$$

where $J_F(x, y)$ represents the Jacobian matrix of F at (x, y).

(b) Show that if $f'(z) = 0$, then $J_F(x, y) = 0$.