King Fahd University of Petroleum and Minerals
Department of Mathematics
MATH533 - Complex Variables
Comprehensive Exam – Term 232

Name: ID:

Time Duration: 3 hrs.

Number of Questions: 7.

3 empty sheets of paper are added for your own sake.

Justify your answers thoroughly. Any theorem that you use must be quoted correctly.

1. Characterize an analytic function f on $\mathbb{C} \setminus \{0\}$ such that

   - f has a pole of order 2 at 0.
   - $\lim_{z\to\infty} f(z) = \infty$.

2. Evaluate the real improper integral

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx,$$

where log means the natural logarithmic function.

3. Let $\Omega = \{z \in \mathbb{C} : \operatorname{Im} z > -1\}$.

(a) Find a Möbius transform which maps Ω onto the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

(b) Show that any analytic function $f : \Delta \setminus \{0\} \to \Omega$ has a removable singularity at 0.

4. Let Δ be the unit disc in $\mathbb{C}$.

(a) Show that for any $a \in \Delta$ and $c \in \mathbb{C}$ with $|c| = 1$,

$$\varphi(z) = c\frac{z-a}{1-\bar{a}z}$$

is an automorphsm of Δ.

(b) Suppose $f \in \mathrm{Aut}(\Delta)$ such that $f(0) = 0$. Show that $f(z) = cz$ for all $z \in \Delta$, for some $c \in \mathbb{C}$ with $|c| = 1$.

(c) Using (a) and (b), show that

$$\mathrm{Aut}(\Delta) = \left\{ c\frac{z-a}{1-\bar{a}z} : a \in \Delta,\ c \in \mathbb{C},\ |c| = 1 \right\}.$$

5. Let $f(z) = z^7 - 5z^5 + 7$. Prove that f has

(a) 5 zeros in $A_1 = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

(b) 2 zeros in $A_2 = \{z \in \mathbb{C} : 2 < |z| < 3\}$.

6. Let U be a domain in $\mathbb{C}$ and $f : U \to \mathbb{C}$ an analytc function. Let $z_0 \in U$.

(a) Prove if $f'(z_0) = 0$, then f is NOT 1-1 on $B(z_0; r)$ for any $r > 0$ such that $\overline{B(z_0; r)} \subset U$, where $B(z_0; r) = \{z \in \mathbb{C} : |z - z_0| < r\}$.

(b) Prove that if $f'(z_0) \neq 0$, then f is 1-1 on $B(z_0; r)$ for sufficiently small $r > 0$ and for $w \in f(B(z_0; r))$,

$$f^{-1}(w) = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{z f'(z)}{f(z) - w} \, dz.$$

7. For a compact set K in $\mathbb{C}$, let

$$\widehat{K} = \{z \in \mathbb{C} : |f(z)| \le \max_{w \in K} |f(w)| \text{ for all entire function } f\}.$$

A domain U in $\mathbb{C}$ is said to be *polynomially convex* if $\widehat{K} \subset U$ whenever K is a compact subset of U. Prove that the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$ is NOT polynomially convex. (Hint. Let $K = \{z \in \mathbb{C} : |z| = 3/2\}$. What is $\widehat{K}$?) .