King Fahd University of Petroleum and Minerals,

Department of Mathematics and Statistics

Comprehensive Exam: Linear Algebra (211)

Duration : 3 Hours

Solve the following questions (**show full details**).

Exercise 1 (20 points: 5-5-5-5) Let T be the linear operator on $\mathbb{R}^3$ defined by $T(x, y, z) = (x, x + y, x + z)$.
(1) Find the matrix $[T]_S$ representing T in the standard basis S of $\mathbb{R}^3$.
(2) Find the matrix $[T]_B$ representing T in the ordered basis $B = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ of $\mathbb{R}^3$.
(3) Find the characterestic and minimal polynomials of T.
(4) Show that T is invertible and find T^{-1} (express T^{-1} explicitely).

Exercise 2 (20 points: 5-5-5-5) Let A and B be $n \times n$ **complex matrices**.
(1) Show that AB and BA have the same nonzero eigenvalues.
(2) Assume that $AB = 0$. Show that $rank(A) + rank(B) \leq n$.
Assume that $A^s = I$ for some positive integer s and let J be the Jordan normal form of A.
(a) Show that $J^s = I$.
(b) Show that A is diagonalizable.

Exercise 3 (20 points:5-5-5-5) Let V be an n-dimensional vector space over the real field $\mathbb{R}$ with a positive definite scalar product $(|)$ and let T be a linear operator on V such that $T^2 = T$ and $TT^t = T^tT$, where T^t is the transpose of T.
(1) Prove that $V = ker(T) \bigoplus Im(T)$.
(2) Prove that $ker(T) = ker(T^t)$.
(3) Prove that for every $v \in V$, $v = w + T(u)$ where $w \in ker(T)$, $||T(v)||^2 = ||T(u)||^2 = (T(v)|T^t(v))$.
(4) Prove that $T = T^t$.

Exercise 4 (20 points)

Find all possible rational forms and their respective Jordan forms of a matrix with characteristic polynomial $(x-1)^3\,(x-2)^4$ and minimal polynomial $(x-1)^2\,(x-2)^2$.

Exercise 5 (20 points: 5-5-5-5)

Let V be a finite dimensional inner product space over a field K and T a self adjoint linear operator on V.

(1) Prove that each eigenvalue of T is real.

(2) Prove that the eigenvectors of T associated with distinct eigenvalues are orthogonal.

(3) Assume that $K = \mathbb{R}$ and let A be a linear on V such that $(Av|v) = 0$ for all $v \in V$. Prove that $A + A^t = 0$ and if A is symmetric, then $A = 0$.

(4) Find a linear operator A such that $(Av|v) = 0$ for all $v \in V$ but $A \neq 0$.