King Fahd University of Petroleum and Minerals
College of Computing and Mathematics
Department of Mathematics

Written Comprehensive Exam (Term 212)
Linear Algebra (Duration = 3 hours)

Problem 1. Let $A := \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{pmatrix}$ on $\mathbb{R}$. Determine its

(1) Characteristic polynomial f
(2) Minimal polynomial p
(3) Jordan form J
(4) Let T be a linear operator on $\mathbb{R}^3$ such that A is the matrix associated to T in the standard basis $\{e_1, e_2, e_3\}$. Show that T has a cyclic vector.

Problem 2. Let $A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$ on $\mathbb{R}$. Determine its

(1) Invariant factors $p_1, \ldots, p_r$
(2) Rational form R
(3) Let T be a linear operator on $\mathbb{R}^3$ such that A is the matrix associated to T in the standard basis $\{e_1, e_2, e_3\}$. Find an explicit cyclic decomposition of $\mathbb{R}^3$ under T; namely, find α , $\beta \in \mathbb{R}^3$ and their respective T-annihilators such that $\mathbb{R}^3 = \mathrm{Z}(\alpha, T) \oplus \mathrm{Z}(\beta, T)$.

Problem 3. **(1)** Let V be the $\mathbb{R}$-vector space of polynomials of degree ≤ 3, endowed with the inner product $(f \mid g) = \int_{-1}^{1} f(t)g(t)dt$. Let W be the subspace spanned by the monomial x^2 (i.e., $W = \mathbb{R}x^2$) and E the orthogonal projection of V on W. Let $f = a + bx + cx^2 + dx^3 \in V$. Find $E(f)$.

(2) Let V be the $\mathbb{R}$-vector space of real-valued continuous functions on the interval $[-1, 1]$, endowed with the inner product $(f \mid g) = \int_{-1}^{1} f(t)g(t)dt$. Find the orthogonal complement of the subspace of even functions.

Problem 4. Let V be a finite-dimensional vector space over $\mathbb{R}$ and let L_1 and L_2 be two *nonzero* linear functionals on V. Consider the bilinear form on V given by

$$f(\alpha,\beta) = L_1\alpha \, L_2\beta$$

(1) Show that $\operatorname{rank}(f) = 1$.

Next, let $V = \mathbb{R}^3$ and let

$$L_1 : \begin{array}{ccc} V & \longrightarrow & \mathbb{R} \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \mapsto & x-y \end{array} \quad ; \quad L_2 : \begin{array}{ccc} V & \longrightarrow & \mathbb{R} \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \mapsto & x-z \end{array}$$

(2) Find the matrix of f in the standard ordered basis $S := \{e_1, e_2, e_3\}$.

(3) Let $B := \{\alpha_1, \alpha_2, \alpha_3\}$ be an ordered basis for V such that the transition matrix from B to S is

$$P = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

Find the matrix of f in B

(4) Is f *non-degenerate* ? (Justify)

Problem 5. Let V be a finite-dimensional vector space over a field F and T a linear operator on V. Let $p = p_1^{r_1} p_2^{r_2} \ldots p_k^{r_k}$ be the minimal polynomial of T, where $r_i \geq 1$ for each i and $p_1, p_2, \ldots, p_k$ are distinct monic irreducible polynomials in $F[x]$. For each $i = 1, \ldots, k$, set $W_i := \operatorname{Nullspace}\left(p_i^{r_i}(T)\right)$.

(1) Announce the Primary Decomposition Theorem.

(2) For each i, prove there exists $\alpha_i \in W_i$ such that the T-annihilator of α_i is equal to $p_i^{r_i}$.

(3) Use (2) to prove there exists $\alpha \in V$ such that the T-annihilator of α is equal to p.