King Fahd University of Petroleum and Minerals
College of Computing and Mathematics
Department of Mathematics

Written Comprehensive Exam (Term 221)

Linear Algebra (Duration = 2 hours)

Name: ———————————————————- ID number: ——————————

Problem 1. Let T_1, T_2, and T_3 be linear operators on $\mathbb{R}^3$ represented in the standard basis $\{e_1, e_2, e_3\}$, respectively, by the following matrices

$$A_1 := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix} \quad ; \quad A_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad ; \quad A_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$$

For each T_i , find

(a) the characteristic polynomial,
(b) the characteristic values,
(c) the minimal polynomial.
(d) Determine if it is diagonalizable. If affirmative, find a basis B such that $[T_i]_B$ is diagonal.
(e) Find a cyclic decomposition of $\mathbb{R}^3$ under T_i

Problem 2. Let M be a matrix with characteristic polynomial $f = x^3(x-1)^4$ and minimal polynomial $p = x^2(x-1)^2$. Find for M all possible rational forms and their respective Jordan forms

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Problem 3. Let $V = F^{n\times n}$ be the vector space of $n \times n$ matrices over a field F and let $B \in V$.

(a) Show that the function f_B defined on V by $f_B(A) = \text{trace}(B^t A)$ is a linear functional.
(b) Show that every linear functional f on V is of the form $f = f_B$ for some B in V.
(c) Show that the mapping $\phi : V \longrightarrow V^\star$, $A \mapsto f_A$ is an isomorphism.

Problem 4. Let V be a finite-dimensional vector space over a field F of characteristic 0 and let f be a *symmetric* bilinear form on V. For each subspace W of V, let $W^\perp$ be the subspace of all vectors α in V such that $f(\alpha, \beta) = 0$ for every β in W. Show that

(a) f is non-degenerate if and only if $V^\perp = 0$
(b) $\text{rank}(f) = \dim(V) - \dim(V^\perp)$.
(c) $\dim(W^\perp) \geq \dim(V) - \dim(W)$.
(d) $V = W \oplus W^\perp$ if and only if the restriction of f to W is non-degenerate.