King Fahd University of Petroleum and Minerals
College of Computing and Mathematics
Department of Mathematics

Written Comprehensive Exam (Term 232)

Linear Algebra (Duration = 3 hours | Max. Score = 100)

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Problem 1. [15]

Let T be the linear operator on $\mathbb{R}^2$ defined by $T(x_1, x_2) = (-x_2, x_1)$

(1) Let the ordered basis $B = \{\alpha_1, \alpha_2\}$, $\alpha_1 = (1,2)$ and $\alpha_2 = (1,-1)$. Find $[T]_B$ the matrix of T in B.

(2) Prove that if B is **any** ordered basis for $\mathbb{R}^2$ and $[T]_B = (a_{ij})$, then $a_{12}a_{21} \neq 0$.

(3) Let W be a nonzero proper subspace of $\mathbb{R}^2$. Prove that W is NOT T-invariant.

Problem 2. [15]

Let W be the space of $n \times n$ matrices over a field F and let W_0 be the subspace spanned by the matrices C of the form $C := AB - BA$. Recall that the trace of an $n \times n$ matrix is equal to the sum of the n entries in the diagonal and let W_1 denote the nullspace of the trace function on W.

(1) Show that $W_0 \subseteq W_1$.

(2) Construct in W_0 a linearly independent set of $n^2 - 1$ elements.

(3) Deduce that W_0 is exactly the subspace of matrices which have trace zero.

`[Use the fact that the trace` $tr : W \longrightarrow F$ `is a linear functional]`

Problem 3. [20]

Find the characteristic and minimal polynomials for each one of the following matrices, and indicate whether the matrix is diagonalizable or not with justification.

(1) $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ over $\mathbb{R}$

(2) $A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ over $\mathbb{R}$

(3) $A = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ over $\mathbb{C}$

Problem 4. [10]

Consider the real matrices $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$, $r \neq 0$.

(1) **Explain** why there must exist a 2×2 invertible matrix P such that $P^{-1}AP$ and $P^{-1}BP$ are both diagonal.

(2) **Find** all matrices P satisfying (1).

Problem 5. [25]

Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(1) Reduce $xI - A$ to its Smith normal form.

(2) Use the Smith normal form of A to find its invariant factors.

(3) Find the rational form for A.

(4) Find the Jordan form for A.

(5) Let T be a linear operator on $\mathbb{R}^4$ such that A is the matrix associated to T in the standard basis $\{e_1, e_2, e_3, e_4\}$.

  (a) Find the respective T-annihilators of e_1, e_2, e_3, and e_4.

  (b) Announce the Cyclic Decomposition Theorem for this case.

  (c) Find an explicit cyclic decomposition of $\mathbb{R}^4$ under T.

Problem 6. [15]

Let V be a finite-dimensional inner product space.

(1) Prove that any orthogonal projection of V on a subspace W is self-adjoint.

(2) Let E be a projection (i.e., $E = E^2$). Prove that if E is normal, then E and $E^\star$ have the same nullspace, and use this fact to show that $V = \text{Range}(E) \bigoplus (\text{Range}(E))^\perp$

(3) Deduce from (1) and (2), that *"a projection is normal if and only if it is self-adjoint."*