KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DEPARTMENT OF MATHEMATICS

Comprehensive Exam, Spring Semester (T242)

Duration: 180 minutes

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Course Code:	MATH 551	**Course Title:**	Abstract Algebra	**Attempt:**	
Name:			**KFUPM ID:**		

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Question	**Max. Grade**	**Grade of the Student**	**Comments (if any)**
Part I : Solve 4 out of the 6 questions in this part			
1	**20**		
2	**20**		
3	**20**		
4	**20**		
5	**20**		
6	**20**		
Part II: Solve **ALL** questions in this part			
7	**8**		
8	**12**		
TOTAL	**100**		
Recommendation			

King Fahd University of Petroleum & Minerals
Department of Mathematics
Comprehensive Exam
Term 241, MATH 551
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Part I: Solve 4 of the 6 questions in this part showing full details.

Q1. (20 points) Let R be a commutative ring, X an indeterminate over R and I a nonzero ideal of R. Consider the polynomial ring $R[X]$ and the ideal

$$I[X] := \{a_0 + a_1X + \cdots + a_nX^n \mid a_i \in I \text{ for each } i,\ n \geq 0 \text{ an integer}\}.$$

(a) Show that if P is a prime ideal of R, then $P[X]$ is a prime ideal of $R[X]$.
(b) Let M be a maximal ideal of R. Is $M[X]$ a maximal ideal of $R[X]$?
(c) Prove that if Q is a P-primary ideal of R (for some prime ideal P of R), then $Q[X]$ is a $P[X]$-primary ideal of $R[X]$.
(d) Find a primary decomposition of the ideal $(4X^2 - 4, 2X^3 - 2X, X^4 - X^2)$ of $\mathbb{Z}[X]$.

Q2. (20 points) Let R be a ring and consider the commutative diagram of left R-modules and R-linear maps such that *each row is exact*:

$$\begin{array}{ccccccc} & & L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 & \longrightarrow 0 \\ & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & \\ 0 & \longrightarrow & L_2 & \xrightarrow[f_2]{} & M_2 & \xrightarrow[g_2]{} & N_2 & \end{array}$$

Prove that:
(a) If α and γ are monomorphisms, then β is a monomorphism.
(b) If α and γ are epimorphisms, then β is an epimorphism.
(c) If α is an epimorphism and β is a monomorphism, then γ is a monomorphism.
(d) If β is an epimorphism and γ is a monomorphism, then α is an epimorphism.

Q3. (20 points) Let R be an integral domain. An R-module M is *divisible* when the equation $rx = m$ has a solution $x \in M$ for every $r \in R\backslash\{0\}$ and $m \in M$.
(a) Show that every injective R-module is divisible.
(b) Show that if R is a PID, then every divisible R-module is injective. (**Hint:** Use *Baer's criterion* of injectivity).
(c) Show that the direct product $\mathbb{Q}^{\Lambda}$ and the direct sum $\mathbb{Q}^{(\Lambda)}$ are injective as Abelian groups for any arbitrary (not necessarily finite) nonempty index set Λ.
(d) Let $\mathbb{P}$ be the set of prime positive integers. For every $p \in \mathbb{P}$, consider the Prüfer group $\mathbb{Z}_{p^\infty}$, i.e. the additive group generated by

$$\{a_1, \cdots, a_n, \cdots \mid\ pa_1 = 0, \cdots, pa_n = a_{n-1} \text{ for } n > 1\}.$$

Show that $\bigoplus_{p\in\mathbb{P}} \mathbb{Z}_{p^\infty}$ is injective as an Abelian group.

Q4. (20 points) Let R be a ring and $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ be an exact sequence of left R-modules.

(a) Show that if $M \simeq L \oplus N$ and $_RM$ is injective, then $_RL$ and $_RN$ are injective.
(b) Show that if $_RL$ is injective, then the sequence splits.
(c) Show that if $_RL$ and $_RM$ are injective, then $_RN$ is injective.
(d) Let $_RM$ and $_RN$ be injective. Is $_RL$ injective?

Q5. (20 points) Let R be an Artinian commutative ring.

(a) Prove that if R is an integral domain, then R is a field.
(b) Show that every prime ideal of R is maximal.
(c) Prove that R has only finitely many prime ideals.
(d) Let N be the *nil radical* of R (i.e. the ideal of all nilpotent elements of R). Prove that N is nilpotent (i.e. $N^k = 0$ for some positive integer k).

Q6. (20 points) Let R be a ring, M a left R-module, L an R-submodule of M and set $N := M/L$. Show that

(a) If $_RM$ is Noetherian, then $_RL$ and $_RN$ are Noetherian.
(b) If $_RL$ and $_RN$ are Noetherian, then $_RM$ is Noetherian.
(c) If $_RM$ is Artinian, then $_RL$ and $_RN$ are Artinian.
(d) If $_RL$ and $_RN$ are Artinian, then $_RM$ is Artinian.

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Part II: Solve each of the following two questions:

Q7. (8 points) Compute the following Abelian groups (*up to isomorphism*):

(a) $\mathbb{Z}_{12} \otimes_{\mathbb{Z}} \mathbb{Z}_{30}$

(b) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$

Q8. (12 points) Prove or disprove (showing full details):

(a) Every commutative Noetherian ring R with zero Jacobson radical is semisimple.

(b) Every left semisimple ring is right semisimple.

(c) If R is a Noetherian commutative ring, then every proper ideal of R has a unique primary decomposition.

GOOD LUCK