King Fahd University of Petroleum and Minerals
Department of Mathematics & Statistics
ODE Comprehensive Exam
The Second Semester of 2020-2021 (202)
Time Allowed: 120mn

Name: ID number:

This is a closed book exam

Problem #	Marks	Maximum Marks
1		25
2		25
3		25
4		25
5		25
Total		$25 \times 4 = 100$

Solve only 4 problems of your choice.

Remark: **In case a student solves all 5 problems, only the first 4 on the exam sheets will be graded.**

Problem 1:(25pts)

1.)(12pts) Find the explicit solution of the IVP. Give the largest interval of definition of the solution.

$$\frac{dy}{dx} = (y^2 - 1)x, \quad x, y \in \mathbb{R},$$
$$y(0) = y_0.$$

2.)(13pts) Show that the IVP has a unique solution in some interval around $x = 0$.

$$\frac{dy}{dx} = -\frac{1}{(y - x)^2},$$
$$y(0) = 1.$$

Problem 2:(25pts) Consider the system

$$\frac{dx}{dt} = y(x^2 + y^2),$$
$$\frac{dy}{dt} = -x(x^2 + y^2),$$
$$\frac{dz}{dt} = z - x^2 - y^2.$$

1.)(4pts) Verify that $X_1(t) = (\sin t, \cos t, 1)$ and $X_2(t) = (0, 0, e^t)$ are solutions of the system.
2.)(10pts) Write the linearized system at the periodic solution $X_1(t)$.
3.)(6pts) Find all characteristics multipliers of the linearized system at $X_1(t)$.
4.)(5pts) Deduce the stability of the periodic solution $X_1(t)$.

Problem 3:(25pts)
Consider the nonlinear system

$$\frac{dx}{dt} = -x - y + 2x(x^2 + y^2),$$
$$\frac{dy}{dt} = x - y + y(x^2 + y^2).$$

1)(8pts) Show that the system has no periodic solution inside the region
$R = \{(x, y) \in \mathbb{R}^2,\ 7x^2 + 5y^2 \leq 1\}$.

2.)(12pts) Give all possible values of $a > 0$ and $b > 0$ such that the bounded set

$$D = \{(x, y) \in \mathbb{R}^2,\ a^2 \leq x^2 + y^2 \leq b^2\}$$

is a trapping region of the system (that is, when a trajectory enters D it remains in D forever, or when a trajectory leaves D it will never return to D forever).

3.)(5pts) Deduce that the system has at least one closed orbit.

Problem 4:(25pts) Let y, f and F be three scalar continuous functions on $\mathbb{R}$. Consider the first order differential equation

$$\frac{dy}{dt} + \frac{1}{t+1} y = F(y), \quad t \geq 0. \tag{1}$$

Assume that $|F(y)| \leq \gamma |y|$ and $|F(y_1) - F(y_2)| \leq \gamma |y_1 - y_2|$, for some $\gamma > 0$.
1.)(5pts) Multiplying Equation (1) by an integrating factor, show that

$$y(t) = \frac{y(0)}{t+1} + \int_0^t \frac{r+1}{t+1} F(y(r)) dr, \ \forall t \geq 0.$$

2.)(10pts) Show that

$$|y(t)| \leq |y(0)| e^{\gamma t}, \ \ \forall t \geq 0.$$

3.) Consider two solutions y_1 and y_2 of Equation (1) such that $y_1(0) = y_2(0)$.
a.)(3pts) Write the differential equation satisfied by $v = y_1 - y_2$.
b.)(7pts) Given an arbitrary $T > 0$, show that

$$v(t) = 0, \ \ \forall t \in [0, T].$$

Problem 5:(25pts) Consider the nonlinear system

$$\frac{dx}{dt} = -x(1-x),$$
$$\frac{dy}{dt} = (x^2 + y^2 - \frac{1}{4})y.$$

1.)(5pts) Find all critical points of the system.
1.)(12pts) Use Lyapunov direct method to show that the origin is asymptotically stable.
2.)(8pts) Study the stability of the point $A(1,0)$.