King Fahd University of Petroleum and Minerals
Department of Mathematics
ODE Comprehensive Exam
The First Semester of 2021-2022 (211)

Time Allowed: 120mn

Name: ID number:

This is a closed book exam

Problem #	Marks	Maximum Marks
1		25
2		25
3		25
4		25
5		25
Total		$25 \times 4 = 100$

<u>Solve only 4 problems of your choice.</u>

<u>Remark:</u> In case a student solves all 5 problems, only the first 4 on the exam sheets will be graded.

Problem 1:(25pts)

1.)(12pts) Find the explicit solution of the IVP

$$\frac{dy}{dx} = y(y+1), \quad x, y \in \mathbb{R},$$
$$y(0) = 1,$$

and indicate the largest interval of definition of this solution.

2.)(13pts) Show that the IVP

$$\frac{dy}{dx} = -\frac{1}{(y - x^2 + 1)^2},$$
$$y(0) = 0,$$

has a unique solution in some interval around $x = 0$.

Problem 2:(25pts) Consider the periodic system

$$\frac{dx}{dt} = -y(x^2+y^2)^2,$$
$$\frac{dy}{dt} = x(x^2+y^2)^2$$
$$\frac{dz}{dt} = 2z - 4(x^2+y^2).$$

1.)(10pts) Write the linearized system at the periodic solution $X_1(t) = (\cos t, \sin t, 2)$.
2.)(10pts) Find all the characteristics multipliers of the linearized system at $X_1(t)$.
3.)(5pts) Deduce the stability of the periodic solution $X_1(t)$.

Problem 3:(25pts)
Consider the nonlinear system

$$\frac{dx}{dt} = 4x + y - 3x(x^2 + y^2),$$
$$\frac{dy}{dt} = -x + 4y - 4y(x^2 + y^2).$$

1)(9pts) Say whether it is possible or not to have a periodic solution of the system inside the region $R = \{(x, y) \in \mathbb{R}^2,\ 13x^2 + 15y^2 \leq 1\}$? Justify your answer.
2.) Let the function $V(x, y) = x^2 + y^2$ and consider the sets

$$\mathcal{C}_1 = \{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 = 1\}$$
$$\mathcal{C}_2 = \{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 = 4\}$$
$$D = \{(x, y) \in \mathbb{R}^2,\ 1 \leq x^2 + y^2 \leq 4\}.$$

a.)(4pts) Show that $\frac{d}{dt}V(x, y) = 2(x^2 + y^2)(4 - 3x^2 - 4y^2)$.
b.)(4pts) Show that $\frac{d}{dt}V(x, y) \leq 0$, for all $(x, y) \in \mathcal{C}_2$.
c.)(4pts) Show that $\frac{d}{dt}V(x, y) \geq 0$ for all $(x, y) \in \mathcal{C}_1$.
d.)(4pts) Admitting that D is a trapping region (also called Bendixson region), what conclusion can we draw from the Poincaré Bindixson theorem?

Problem 4:(25pts) Let y be a positive scalar continuous function on $[0, \infty)$. Consider the first order differential inequality

$$\frac{dy}{dt} + 2y^2 = 4y + 1, \qquad t \geq 0. \tag{1}$$

1.)(5pts) Using the Young inequality, deduce from Equation (1) that

$$\frac{dy}{dt} + y^2 \leq 5, \qquad \forall t \geq 0. \tag{2}$$

2.)(10pts) Multiplying (2) by an integrating factor, show that, if $\int_0^t y(s)ds \geq \alpha$, then

$$y(t) \leq y(0)e^{-\alpha} + 5t, \ \ \forall t \geq 0.$$

3.) Consider two solutions y_1 and y_2 of Equation (1) such that $y_1^2 - y_2^2 \geq 0$ and $y_1(0) = y_2(0)$.
a.)(4pts) Write the differential equation satisfied by $v = y_1 - y_2$.
b.)(6pts) Given an arbitrary $T > 0$, show that

$$v(t) = 0, \ \ \forall t \in [0, T].$$

Problem 5:(25pts) Consider the nonlinear system

$$\frac{dx}{dt} = x(x^2 + y^2 - 1),$$
$$\frac{dy}{dt} = -y(1 - y^2).$$

1.)(6pts) Find all the critical points of the system.
2.)(7pts) Study the stability of the critical point $C(1, 0)$.
3.) Let the function $V(x, y) = x^2 + y^2$ and the disk $D = \{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 \leq \frac{1}{4}\}$.
a.)(4pts) Compute the derivative $\frac{d}{dt}V(x, y)$.
b.)(4pts) Show that $\frac{d}{dt}V(x, y) < 0$, for all $(x, y) \in D$.
c.)(4pts) Deduce the stability of the origin.