King Fahd University of Petroleum and Minerals
Department of Mathematics
ODE Comprehensive Exam
The Second Semester of 2022-2023 (222)
Time Allowed: 150min

Name: ID number:

This is a closed book exam

Problem #	Marks	Maximum Marks
1		25
2		25
3		25
4		25
5		25
Total		$25 \times 4 = 100$

Solve only 4 problems of your choice.

Remark: In case a student solves all 5 problems, only the first 4 on the exam sheets will be graded.

Problem 1:

1.)(12pts) Consider the IVP

$$\frac{dy}{dx} = (2y-1)(y-1),$$
$$y(0) = \frac{1}{3}.$$

Find the explicit solution of this IVP and indicate the largest interval of definition of the solution.

2.)(13pts) Show that the IVP

$$\frac{dy}{dx} = \sqrt{1-y} + \sqrt{1-x^2},$$
$$y(0) = \frac{1}{2},$$

has a unique solution in some interval around $x = 0$, and give this interval.

<u>**Solution:**</u> .

Problem 2: Consider the nonlinear system

$$\frac{dx}{dt} = -2y(x^2+y^2),$$
$$\frac{dy}{dt} = 2x$$
$$\frac{dz}{dt} = 3z + 3(x^2+y^2).$$

1.)(5pts) Show that system has a periodic solution of the form $X_1(t) = (\alpha \cos \omega t, \alpha \sin \omega t, \beta)$, for some $\alpha, \omega > 0$ and $\beta \in \mathbb{R}$.
2.)(20pts) Analyze the stability of the periodic solution $X_1(t)$ by using the Floquet theory.

Solution:

Problem 3: Consider the nonlinear system

$$\frac{dx}{dt} = x(x^2 + y^2 - 3x - 4) - y,$$
$$\frac{dy}{dt} = y(x^2 + y^2 - 3x - 4) + x.$$

1)(5pts) Find a region in the xy-plane, where it is not possible to have a periodic solution.
2.)(20pts) Find a trapping region, where we have the guarantee that there exists a periodic solution of the system.

<u>**Solution:**</u>

Problem 4: Consider the first order differential equation

$$\frac{dy}{dt} + y = -4y^3 + 1, \qquad t \geq 0. \tag{1}$$

1.)(5pts) Show that $y^2(t) \leq y^2(0) + 1$, $\forall t \geq 0$.
2.)(4pts) Show that $\int_0^t y^2(s)ds \leq y^2(0) + t$, $\forall t \geq 0$.
3.)(16pts) Show that Problem (1) cannot have more than two different solutions with the same initial condition $y(0)$.

<u>**Solution:**</u>

Problem 5: Consider the nonlinear system

$$\frac{dx}{dt} = x(y^2 - 4),$$
$$\frac{dy}{dt} = -y(1 - x^2).$$

1.)(8pts) Find all the critical points of the system.
2.)(6pts) Study the stability of the critical point $C(1, 2)$.
3.)(8pts) Find a closed domain of $\mathbb{R}^2$ containing no other critical point than the origin, and a positive definite function $V(x, y)$ such that $\frac{dV}{dt}(x, y)$ is negative definite on D.
4.)(3pts) Deduce the stability of the origin.

<u>**Solution:**</u>