King Fahd University of Petroleum and Minerals
Department of Mathematics
ODE Comprehensive Exam
The First Semester of 2023-2024 (231)
Time Allowed: 180min

Name: ID number:

This is a closed book exam

Problem #	Marks	Maximum Marks
1		25
2		25
3		25
4		25
5		25
Total		$25 \times 4 = 100$

Solve only 4 problems of your choice.

Remark: In case a student solves all 5 problems, only the first 4 on the exam sheets will be graded.

Problem 1:

1.) Consider the IVP

$$(x^2-4)\sin^3 y dy = \frac{1}{\cos^2 y}dx,$$
$$y(0) = 0.$$

a.)(9pts) Find an implicit solution of the IVP. Write this solution without an absolute value.

b.)(4pts) Indicate the largest interval of definition of this solution.

2.) Consider the IVP

$$\frac{dy}{dx} = (x^2-2)y^{\frac{2}{3}} + 2,$$
$$y(1) = 0.$$

a.)(8pts) Show that the IVP has a solution in some interval to be given explicitly.

b.)(4pts) Is this solution unique? Justify your answer.

Solution: .

Problem 2:

1.)(13pts) Find a periodic solution of the system

$$\begin{aligned}\frac{dx}{dt} &= 2\pi y,\\ \frac{dy}{dt} &= -2\pi x(x^2+y^2),\end{aligned}$$

and analyze its stability.

3.)(12pts) Find the characteristic multipliers of the system

$$X' = \begin{pmatrix} -1 & 1 \\ 0 & 1+\cos t - \frac{\sin t}{2+\cos t} \end{pmatrix} X.$$

Solution:

Problem 3:
Consider the nonlinear system

$$\frac{dx}{dt} = -x(x^2 - xy + y^2) + 2x,$$
$$\frac{dy}{dt} = -3y(x^2 - xy + y^2) + 2(y - x).$$

1)(10pts) Consider the ellipse

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2,\ 3x^2 - 4xy + 5y^2 = 2\}.$$

If the system has a periodic solution, does the orbit of this periodic solution crosses $\mathcal{C}$ or no? Justify your response clearly.
2.)(15pts) Prove that the systems has a periodic solution.

Solution:

Problem 4:

1.)(13pts) Consider the ODE

$$\frac{dy}{dx} + y = 25y^3 - 1, \quad x \in (-\infty, 0].$$

Show that

$$y^2(t) \leq y^2(0)e^{-t} - 1 + e^{-t}, \ \forall t \leq 0.$$

2.)(12pts) Consider the ODE

$$\frac{dy}{dx} + y^9 = 3y^3, \quad x \in [0, \infty).$$

Given that $\int_0^t y^2(x)dx \leq 3$, $\forall t \geq 0$, show that

$$y^2(t) \leq y^2(0)e^{18}, \ \forall t \geq 0.$$

Solution:

Problem 5:

1.)(12pts) Consider the nonlinear system

$$\frac{dx}{dt} = \sqrt{x}(x^2 - y),$$
$$\frac{dy}{dt} = \frac{1}{y}(1 - x).$$

Study the stability of the critical point A(1,1).

2.)(13pts) Consider the nonlinear system

$$\frac{dx}{dt} = -x - 3y,$$
$$\frac{dy}{dt} = 3x.$$

Show that the origin is globally asymptotically stable.

Solution: