King Fahd University of Petroleum and Minerals
Department of Mathematics

ODE Comprehensive Exam

The Second Semester of 2023-2024 (232)

Time Allowed: 180min

Name: ID number:

This is a closed book exam

Problem #	Marks	Maximum Marks
1		25
2		25
3		25
4		25
5		25
Total		$25 \times 4 = 100$

Solve only 4 problems of your choice.

Remark: In case a student solves all 5 problems, only the first 4 on the exam sheets will be graded.

Problem 1:

1.)Consider the IVP

$$\frac{dy}{dx} = y^2 - 4,$$
$$y(0) = 1.$$

a.)(9pts) Find an explicit solution of the IVP.
b.)(4pts) Indicate the largest interval of definition of this solution.
2.) Consider the IVP

$$\frac{dy}{dx} = \sqrt{1 - x^2} + y^2,$$
$$y(0) = 0.$$

a.)(8pts) Show that the IVP has a solution in some interval I to be given explicitly.
b.)(4pts) Is this solution unique? Justify your answer.

<u>**Solution:**</u> .

Problem 2:

1.)(12pts) Analyze the stability of the periodic solution $X(t) = (\cos t, \sin t)$ of the system

$$\frac{dx}{dt} = -y(x^2 + y^2),$$
$$\frac{dy}{dt} = x(x^2 + y^2),$$

2.)(13pts) Find the characteristic multipliers of the system

$$X' = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 + \cos t & 0 \\ 1 & 1 & 0 \end{pmatrix} X,$$

given that the system has one periodic solution X_1 and another solution $X_2(t) = (2\sin t + \cos t, 1, 0)e^{2t}$.

<u>**Solution:**</u>

Problem 3:

Consider the nonlinear system

$$\frac{dx}{dt} = -6x + 3y + x(x^2 + y^2), \tag{1}$$

$$\frac{dy}{dt} = -3x - 6y + 2y(x^2 + y^2). \tag{2}$$

1)(10pts) Show that the system has no periodic solution inside the region

$$R = \{(x, y) \in \mathbb{R}^2,\ 5x^2 + 7y^2 < 12\}.$$

2.)(15pts) Prove that the systems (1)-(2) has at least one periodic solution.

Solution:

Problem 4:

1.)(12pts) Consider the ODE

$$\frac{dy}{dt} + y = -y^5 + 2, \quad t \in [0, \infty).$$

Show that

$$y^2(t) \leq y^2(0)e^{-t} + 4, \ \forall t \geq 0.$$

2.)(13pts) We assume that the following IVP

$$\frac{dy}{dt} = f(y) + \cos t,$$
$$y(0) = y_0.$$

has a solution $y = y(t)$, $\forall t \in [0, 2]$.
We also assume that the function f satisfies the estimate

$$|f(y_1) - f(y_2)| \leq |y_1 - y_2|.$$

Show that this IVP has a unique solution.

<u>Solution:</u>

Problem 5:

1.) Consider the nonlinear system

$$\frac{dx}{dt} = x(4x^2 - y^2),$$
$$\frac{dy}{dt} = y(1 - x).$$

a.)(4pts) Find all the critical points of the system.

b.)(8pts) Study the stability of the point $A(1, 2)$.

2.) Consider the nonlinear system

$$\frac{dx}{dt} = 3y,$$
$$\frac{dy}{dt} = -2y - 3x.$$

a.)(9pts) Let $V(x, y) = x^2 + xy + y^2$. Compute $\frac{d}{dt}V(x(t), y(t))$.

b.)(4pts) Deduce the stability of the origin from part a.).

Solution: