Department of Mathematics, KFUPM

Fractional Differential Equations

Comprehensive Exam (T232)

Duration: 180 minutes

Student Name: ....

Q	1	2	3	4	5	6	7	8	Total
Max	10	10	10	10	10	10	10	10	70
Points									

Instructions

- **Seven Problems only:** Select any seven problems that you feel confident about solving. Only the first 7 chosen problems will be graded.
- **Separate Sheets:** For each problem in the exam, use separate sheet of papers to write down your solution.
- **Labeling:** At the top of each sheet, clearly write the problem number.
- **Order:** Arrange the sheets in order according to the problem numbers.
- **Points distribution:** the maximum score is 70 points, 10 points for each problem.

1) Compute ${}^{c}D_0^{\alpha} f$, $0 < \alpha < 1$, $t > 0$, for

$$f(t) = \begin{cases} t, & t < 1, \\ 1 - t, & t \geq 1. \end{cases}$$

2) Let $f \in L^1(a,b)$. Consider the Riemann-integral of order $\alpha > 0$,

$$(I_{b-}^{\alpha} f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} f(s)\, ds\,, \qquad x < b.$$

Show that $I_{b-}^{\alpha} f \in AC[a,b]$ for $\alpha > 1$.

3) Let $f \in C[a,\infty)$ and $1 < \alpha < 2$. Show that $u \in AC[0,\infty)$ is a solution of

$$\begin{aligned} {}^{c}D_0^{\alpha} u(t) &= f(t), \qquad t > 0, \\ u(0) &= u_0, \qquad u'(0) = u_1, \end{aligned}$$

if and only if u is a solution of the problem

$$\begin{aligned} u'(t) &= I_0^{\alpha-1} f(t) + u_1, \\ u(0) &= u_0. \end{aligned}$$

4) Solve the Cauchy problem

$$\begin{aligned} &{}^{c}D_0^{8/3} y(t) - 4y(t) = 0, \qquad t > 0, \\ &y(0) = 1, \qquad y'(0) = 0, \qquad y''(0) = 2. \end{aligned}$$

5) Prove the following identity **or** show it is not correct,

$$ {}^{RL}D_0^{1/2}\, {}^{RL}D_0^{3/2} E_\alpha(x) = D^2 E_\alpha(x). $$

6) Calculate the following derivative,

$$ {}^{c}D_0^{\alpha}(x-1)^{\alpha}, \qquad x>1, \qquad 0<\alpha<1. $$

Hint. $(x+y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k$, $|x|>|y|$.

7) Use successive approximation to compute $u_1(t)$ and $u_2(t)$ only for

$$ {}^{c}D_0^{\alpha}u = t + u^2, \qquad 0<\alpha<1, \qquad t>0, $$
$$ u(0)=0. $$

8) Let $u \in C^1[0,T]$, $T>0$.

(a) Show that for $\alpha \in (0,1]$,

$$ {}^{c}D_0^{\alpha}u^2(t) \le 2\,u(t)\ {}^{c}D_0^{\alpha}u(t) \iff \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{[u(t)-u(s)]\,u'(s)}{(t-s)^{\alpha}}\,ds \ge 0. $$

(b) Give a function u such that ${}_0^cD_t^{\alpha}u^2(t) = 2\,u(t)\ {}^{c}D_0^{\alpha}u(t)$.

(c) Give a function u such that ${}_0^cD_t^{\alpha}u^2(t) < 2\,u(t)\ {}^{c}D_0^{\alpha}u(t)$.

(Hint. Consider $u(t)=t^{\beta}$)

Formula

$$ I_0^{\alpha}\, {}^{RL}D_0^{\alpha} f(t) = f(t) - \sum_{k=1}^{n} \frac{D_0^{\alpha-k} f(0)}{\Gamma(\alpha-k+1)}\, t^{\alpha-k} $$

$$ {}^{c}D_a^{\alpha} f := {}^{RL}D_a^{\alpha}\left[f - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}\,(x-a)^k \right] $$

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} $$

$$ \mathcal{L}\{t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\} = \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}, \qquad \mathcal{L}\{1\} = 1/s $$

$$ \mathcal{L}\{I_0^{\alpha} f(t)\} = s^{-\alpha}F(s) $$

$$ \mathcal{L}\{{}^{RL}D_0^{\alpha} f\} = s^{\alpha}F(s) - \sum_{k=1}^{n} s^{k-1}\left(D_0^{\alpha-k} f\right)(0) $$

$$ \mathcal{L}\{{}^{C}D_0^{\alpha} f\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}\left(D^k f\right)(0) $$
