King Fahd University of Petroleum and Minerals
Department of Mathematics
PDE Comprehensive Exam
The Second Semester of 2021-2022 (212)
Time Allowed: 150mn

Name: ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		25
2		25
3		25
4		25
5		25
Total		25 $\times$4 =100

Solve 4 problems of your choice.

Remark: **Only 4 problems will be taking into account, even if you solve the five problems.**

Problem 1:(25pts)

1.)(13pts) Use the method of characteristics to solve the Cauchy problem:

$$u_x + (1+y^2)u_y = u, \quad x, y \in \mathbb{R}, \tag{1}$$

$$u(0,y) = \tan^{-1} y. \tag{2}$$

2.)(12pts) Find the canonical form of the the parabolic PDE

$$u_{xx} - 2u_{xy} + u_{yy} = u. \tag{3}$$

Solution:

Problem 2:(25pts)
1.)(12pts) Solve the nonhomogeneous problem

$$u_{tt} - 4u_{xx} = x, \quad x \in \mathbb{R},\ t > 0, \tag{4}$$
$$u(x,0) = 0, \quad x \in \mathbb{R}, \tag{5}$$
$$u_t(x,0) = 0, \quad x \in \mathbb{R}. \tag{6}$$

Hint: You may write the problem satisfied by $v = u + \frac{x^3}{24}$ and apply the d'Alembert formula, or you may directly use the formula $u(x,t) = \frac{1}{4}\iint_\Delta X dA$, where Δ is the characteristic triangle at (x,t).
2.)(13pts) Solve the Cauchy problem

$$u_{tt} = u_{xx} + u_{yy} + u_{zz}, \quad (x,y,z) \in \mathbb{R}^3,\ t > 0, \tag{7}$$
$$u(x,y,z,0) = 0, \qquad (x,y,z) \in \mathbb{R}^3, \tag{8}$$
$$u_t(x,y,z,0) = x, \qquad (x,y,z) \in \mathbb{R}^3. \tag{9}$$

Hint: You can use the Kirchhoff formula $u(x,y,z,t) = \frac{1}{4\pi t}\iint_{S_t} u_t(X,Y,Z,0)d\sigma_t$, where S_t is the sphere of center (x,y,z) and radius t, and σ_t is the surface area of S_t.

<u>**Solution:**</u>

Problem 3:(25pts)

1.)(15pts)Solve the initial value problem

$$u_t - u_{xx} = 0, \quad -\infty < x < \infty,\ t > 0. \tag{10}$$

$$u(x,0) = \begin{cases} 1, & \text{if} \quad x \in [-1,1], \\ 0, & \text{elsewhere.} \end{cases} \tag{11}$$

Hint: Look for bounded solutions.

2.)(10pts) Consider the initial value problem

$$u_t - u_{xx} = 0, \quad 0 < x < 1,\ t > 0, \tag{12}$$

$$u(x,0) = 2, \qquad 0 \le x \le 1, \tag{13}$$

$$u(0,t) = u(1,t) = 2, \quad t \ge 0. \tag{14}$$

By using the weak maximum principle, what conclusion can we reach for the solution $u(x,t)$ of this problem? Justify your answer.

Solution:

Problem 4:(25pts)

1.)(13pts) We consider initial and boundary value the problem

$$u_{tt} + u_t - u_{xx} = 0, \quad 0 < x < L, \ \ t > 0 \tag{15}$$

$$u(0,t) = f(t), \ \ u(L,t) = g(t), \quad t \geq 0 \tag{16}$$

$$u(x,0) = \psi(x), \ \ u_t(x,0) = \varphi(x), \quad 0 \leq x \leq L, \tag{17}$$

where f, g, ψ, φ are continuous functions.

Show that the problem has a unique solution.

Hint: You may estimate the quantity $\frac{d}{dt}\int_0^L (w_t^2 + w_x^2)dx$, where w is the difference of two solutions of the problem.

2.)(12pts) Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. Consider the initial and boundary value problem

$$u_t - \Delta u + u^5 = 0, \qquad x \in \Omega, \ \ t > 0, \tag{18}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega, \ \ t \geq 0, \tag{19}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega. \tag{20}$$

Show that, if $\int_\Omega (u_0(x))^2 dx \leq 1$, then

$$\int_\Omega (u(x,t))^2 dx \leq 1, \quad \forall t \geq 0. \tag{21}$$

Solution:

Problem 5:

1.)(5pts) Let $u \in \mathcal{C}^2(\mathbb{R})$ be a solution of the equation

$$u_{xx} + 4u + 4u_x + u_{yy} = 0. \tag{22}$$

Find a twice differentiable function f on $\mathbb{R}$ such that $f(0) = 1$, $f'(0) = 2$ and such that $v(x, y) = f(x)u(x, y)$ is solution of the Laplace equation

$$v_{xx} + v_{yy} = 0. \tag{23}$$

2.)(20pts) Solve the Laplace problem

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < 1,\ 0 < y < 1, \tag{24}$$
$$u_x(0, y) = u_x(1, y) = 0, \quad 0 \le y \le 1, \tag{25}$$
$$u(x, 0) = 0, \quad u(x, 1) = g(x), \quad 0 \le x \le 1. \tag{26}$$

Solution: