King Fahd University of Petroleum and Minerals
Department of Mathematics
PDE Comprehensive Exam
The First Semester of 2022-2023 (221)
Time Allowed: 120mn

Name: ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		25
2		25
3		25
4		25
5		25
Total		25 $\times$4 =100

Solve 4 problems of your choice.

Remark: **Only 4 problems will be taking into account, even if you solve the five problems.**

Problem 1:(25pts)

1.)(13pts) Use the method of characteristics to solve the Cauchy problem:

$$u_x + e^{-y}u_y = e^y, \quad u(0, y) = e^{2y}, \quad x > 0,\ y \in \mathbb{R}.$$

2.)(12pts) Find the general solution $w(\xi, \eta)$ of the hyperbolic PDE

$$w_{\xi\eta} + w_\xi = 2\xi.$$

Solution:

Problem 2:(25pts)

1.)(10pts) Find the solution of the 1D Cauchy problem

$$u_{tt} - 9u_{xx} = 0, \;\; (x,t) \in \mathbb{R} \times (0,\infty), \tag{1}$$
$$u(x,0) = x^2, \;\; u_t(x,0) = \cos^2 x, \;\; x \in \mathbb{R}. \tag{2}$$

2.)(15pts) Solve the 2D Cauchy problem

$$u_{tt} = u_{xx} + u_{yy}, \quad (x,y) \in \mathbb{R}^2, \;\; t > 0, \tag{3}$$
$$u(x,y,0) = 0, \qquad (x,y) \in \mathbb{R}^2, \tag{4}$$
$$u_t(x,y,0) = y, \qquad (x,y) \in \mathbb{R}^2. \tag{5}$$

Solution:

Problem 3:(25pts)

1.)(15pts) Solve the initial and boundary value problem

$$u_t = 4u_{xx}, \quad x \geq 0,\ t \geq 0, \tag{6}$$

$$u(x,0) = \begin{cases} 1, & \text{if} \quad x \in [0,\pi], \\ 0, & \text{elsewhere}, \end{cases} \tag{7}$$

$$u(0,t) = 0, \quad \forall t \geq 0. \tag{8}$$

2.)(10pts) Assume $u(x,t) = \sum_{n=1}^{\infty} 2A_n \sin(n\pi x) e^{-\pi^2 n^2 t}$ is the solution the IBVP

$$u_t = u_{xx}, \quad 0 < x < 1,\ t > 0, \tag{9}$$

$$u(x,0) = 1,\ \ 0 \leq x \leq 1. \tag{10}$$

Compute explicitly A_n.

<u>Solution:</u>

Problem 4:(25pts)

1.)(10pts) Let Ω be a bounded domain of $\mathbb{R}^3$, with smooth boundary $\partial\Omega$.

Consider the boundary value problem

$$-\Delta u + u = \lambda u, \quad \text{on } \Omega, \tag{11}$$
$$u = 0, \quad \text{on } \partial\Omega, \tag{12}$$

where $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

Prove that if this problem admits a non trivial solution u (that is, $u \neq 0$), then $\lambda \geq 1$.

2.)(15pts) Use the weak maximum principle to show that any continuous solution u of the initial and boundary value problem

$$u_t = u_{xx} + 2, \quad 0 < x < 1, \; t > 0 \tag{13}$$
$$u(0,t) = u(1,t) = 2t, \quad t \geq 0, \tag{14}$$
$$u(x,0) = 0, \quad 0 \leq x \leq 1 \tag{15}$$

satisfies the estimate $u(x,t) \leq 2t$, for all $0 \leq x \leq 1$ and $t \geq 0$.

Note: Use the weak maximum principle **only** to answer the question. Any other different method to answer the question will be given a grade of zero.

Solution:

Problem 5:(25pts)
1.)(5pts) Consider the problem

$$u_t + \mu(t)u = \Delta u, \quad x \in \mathbb{R}^3, \ t > 0. \tag{16}$$

Write the differential equation satisfied by the function v, where $v(x,t) = u(x,t)e^{\int \mu(t)dt}$.
2.)(20pts) Solve the 2D Neumann problem

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < 1, \ 0 < y < 2, \tag{17}$$
$$u_x(0,y) = u_x(1,y) = 0, \quad 0 \le y \le 2, \tag{18}$$
$$u_y(x,0) = 0, \quad u_y(x,2) = g(x), \quad 0 \le x \le 1. \tag{19}$$

Solution: