Department of Mathematics and Statistics, KFUPM
Comprehensive Exam, Math 571 (T232)
Duration: 180 mins

Student Name: ....

Instructions

- **Five Problems only:** Select any five problems that you feel confident about solving. Only the first 5 chosen problems will be graded.
- **Separate Sheets:** For each problem in the exam, use separate sheet of papers to write down your solution.
- **Labeling:** At the top of each sheet, clearly write the problem number.
- **Order:** Arrange the sheets in order according to the problem numbers.
- **Points distribution:** the maximum score is 100 points, 20 points for each problem.

Problem 1: Consider the multistep ODE schemes for solving

$$y'(x) = f(y(x), x).$$

(a) Show that the scheme

$$y_n - y_{n-1} = h[\theta f_n + (1-\theta) f_{n-1}]$$

is A-stable for $1/2 \leq \theta \leq 1$.
(b) Consider the scheme

$$y_n - 3y_{n-1} + 2y_{n-2} = -\frac{1}{2}h(f_n + f_{n-1}).$$

Is it consistent? Is it stable?
(c) Consider the scheme

$$y_n - y_{n-2} = h(f_n - 3f_{n-1} + 4f_{n-2}).$$

Is it stable? Compute its order.

Problem 2: The Butcher tableaux for a Runge-Kutta method is given by

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

(a) Is the method explicit or implicit scheme? Justify?

(b) Write down the corresponding scheme for solving $y' = f(x, y(x)),\ \ y(x_0) = y_0$ and determine if it is consistent or not.

(c) Show that the method is a second order method.

Problem 3 : Consider the Cauchy problem $y' = f(x, y(x)),\ \ y(x_0) = y_0$ and the multistep formula

$$\eta_{n+1} = (1-a)\eta_n + a\eta_{n-1} + \frac{h}{12}\{(5-a)f_{n+1} + 8(1+a)f_n + (5a-1)f_{n-1}\}.$$

Here a is a real parameter, $f_n = f(x_n, \eta_n),\ x_n = x_0 + nh$ and η_n is an approximation of $y(x_n)$.

(a) Show that the above method has a local truncation error which is $\mathcal{O}(h^3)$ for all a.

(b) Find a value for a so that the local truncation error is $\mathcal{O}(h^4)$ and check whether the resulting scheme satisfies the root condition.

(c) Define the region of absolute stability of a numerical method for solving the Cauchy problem for an ordinary differential equation. Define what it means for a multistep method to be A–stable. Is the method of Part (b) A–stable.

Problem 4: Consider the two-point boundary-value problem

$$-y''(x) + y'(x) + y(x) = x^2 \quad \text{for } x \in (0,1) \quad \text{with} \quad y(0) = y(1) = 0, \tag{1}$$

(a) Define the finite difference solution of (1) over a uniform mesh consists of N equal-sub-intervals of length h and show that the resulting system is tridiagonal system.

(b) What is the order of the truncation error?

(c) Explain how you would solve the boundary-value problem (1) by the shooting method. You should describe and explain all relevant details, but should not solve any equations.

Problem 5: The following Predictor-Corrector method is to be used for solving $y'(x) = f(x, y(x)),\ \ y(x_0) = y_0$.

$$\text{Predictor: } y_{i+1} = y_i + h\left(\frac{3}{2}f(x_i, y_i) - \frac{1}{2}f(x_{i-1}, y_{i-1})\right)$$
$$\text{Corrector: } y_{i+1} = y_i + h\left(\frac{1}{2}f(x_{i+1}, y_{i+1}) + \frac{1}{2}f(x_i, y_i)\right)$$

(a) Determine the order of the method.

(b) Choose an appropriate method for initiating the method. Justify your answer.

(c) Write an iterative technique for implementing the method.

Problem 6: Consider the initial value problem

$$y'(x) = \ln\left(\ln(4 + y^2)\right), \quad x \in [0, 1],\ y(0) = 1,$$

(a) Write the one-step explicit Euler scheme and show that the truncation error $|T_n| \leq h/4$.

(b) Verify that

$$|y(x_{n+1}) - y_{n+1}| \leq (1 + hL)|y(x_n) - y_n| + h|T_n|, \quad n = 0, \cdots, N-1,$$

where $L = 1/(2\ln 4)$.

(c) Find a positive integer N_0, as small as possible, such that

$$\max_{0 \leq n \leq N} |y(x_n) - y_n| \leq 10^{-4}$$

whenever $N \geq N_0$.

Good luck