PhD Comprehensive Exam

Duration: 180 minutes

ID:	
NAME:	

- Show your work.
- There are empty pages attached to this exam booklet.
- Calculators are allowed.
- You may use the following formulae
  (i) Newton's divided difference formula
  $$p_n(x) = f[x_0] + \sum_{j=1}^{n} f[x_0, \cdots .x_j] \prod_{i=1}^{j-1}(x - x_i),$$
  (ii) Lagrange basis
  $$L_j(x) = \prod_{\substack{1 \le j \le n, \\ k \ne j}} \frac{x - x_k}{x_j - x_k}, \quad j = 1, 2, \ldots, n.$$

Problem	Score
1	
2	
3	
4	
5	
6	
Total	/100

Problem 1 (20 points)

Consider the function
$$f(x) = x^3 + 1.$$

(a) Use $x_0 = 0, x_1 = \frac{1}{2}$ and $x_2 = 1$ to construct the Lagrange interpolation polynomial of degree at most two to approximate $f(\frac{1}{4})$ and find the absolute error of this approximation.

(b) Explain why the obtained absolute error from (a) is given by
$$|\text{error}| = \left| \left(\frac{1}{4} - x_0\right) \left(\frac{1}{4} - x_1\right) \left(\frac{1}{4} - x_2\right) \right|.$$

Problem 2 (25 points)

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and h a positive real number. Define the forward difference operator Δ_h by

$$\Delta_h f(x) = f(x+h) - f(x).$$

Powers of Δ_h are defined recursively by

$$\Delta_h^0 f(x) = f(x), \qquad \Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1} f(x)), \quad k = 1, 2, \ldots$$

(a) Show by induction on n that

$$f[x, x+h, x+2h, \ldots, x+nh] = \frac{1}{n!h^n} \Delta_h^n f(x),$$

where

$$f[x_0, x_1, x_2, \ldots, x_j] = \frac{f[x_1, x_2, \ldots, x_j] - f[x_0, x_1, x_2, \ldots, x_{j-1}]}{x_j - x_0}, \text{ and } f[x_0] = f(x_0).$$

is the divided difference of f at $x_0, x_1, x_2, \ldots, x_j$.

(b) Using the Newton's divided difference interpolation formula, show that the interpolating polynomial of degree at most n of f at the points $x_0, x_0+h, x_0+2h, \ldots, x_0+nh$ is given by the *Newton forward difference formula*

$$p_n(x) = \sum_{j=0}^{n} \binom{s}{j} \Delta_h^j f(x_0)$$

where

$$s = \frac{x - x_0}{h}, \quad \binom{s}{j} = \frac{s(s-1)\ldots(s-j+1)}{j!}, \quad \text{with} \quad \binom{s}{0} = 1.$$

Problem 3 (15 points)

Consider the function

$$f(x) = a^x \quad \text{with} \quad a > 1 \quad \text{a real number.}$$

(a) With the notation of the previous Problem, show by induction on the integer k that

$$\Delta_1^k f(x) = (a-1)^k a^x$$

(b) Conclude, using Newton forward difference interpolation formula, that the interpolating polynomial for f constructed at the integers $0, 1, 2, \ldots, n$, is given by

$$p_n(x) = \sum_{k=0}^{n} (a-1)^k \binom{x}{k}$$

(c) Use part (b) to show the identity

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Problem 4 (15 points)

Let $a < b \in \mathbb{R}$. Let $\{\pi_0, \pi_1, \ldots, \pi_n, \ldots\}$ be a sequence of orthogonal polynomials on the interval $[a, b]$ with respect to a weight function $\omega : [a, b] \to (0, \infty)$, i.e. degree of $\pi_j = j$ and

$$\langle \pi_i, \pi_j \rangle = \int_a^b \pi_i(x)\pi_j(x)\omega(x)dx = 0, \quad i \neq j.$$

Further, let $x_1, x_2, \ldots, x_n$ be the zeros of π_n.

(a) Show that the Lagrange polynomials of degree $(n-1)$ based on $x_1, x_2, \ldots, x_n$ are orthogonal to each other, that is

$$\int_a^b L_i(x)L_j(x)\omega(x)dx = 0, \quad i \neq j,$$

(b) For a given function f, let

$$y_k = f(x_k), \quad k = 1, 2, \ldots, n.$$

Show that the polynomial $p_{n-1}(x)$ of degree at most $(n-1)$ that interpolates the function f at the zeros $x_1, x_2, \ldots, x_n$ satisfies

$$\|p_{n-1}\|^2 = \sum_{k=1}^{n} y_k^2 \|L_k\|^2,$$

where the weighted norm $||.||$ is defined as

$$||g|| = \sqrt{\int_a^b g(x)^2\omega(x)dx},$$

for any suitably integrable function g.

Problem 5 (15 points)

Let S be the cubic spline that interpolates a function $f \in C^2[a,b]$ at the knots

$$a = x_1 < x_2 < \ldots < x_n = b$$

and satisfies the **clamped boundary conditions**

$$S'(a) = f'(a), \quad S'(b) = f'(b).$$

For $x \in [a,b]$, let $D(x) = f(x) - S(x)$,

(a) Using integration by parts show that

$$\int_a^b S''(x)D''(x)dx = -\int_a^b S^{(3)}(x)D'(x)dx.$$

(b) Divide the interval $[a,b]$ into sub-intervals and use integration by parts to show that

$$\int_a^b S^{(3)}(x)D'(x)dx = 0.$$

(c) Conclude that

$$\int_a^b \left[S''(x)\right]^2 dx \leq \int_a^b \left[f''(x)\right]^2 dx.$$

Problem 6 (10 points)

Let $a < b \in \mathbb{R}$. Let $\{\pi_0, \pi_1, \ldots, \pi_n, \ldots\}$ be a sequence of **orthonormal** polynomials on the interval $[a, b]$ with respect to a weight function $\omega : [a, b] \to (0, \infty)$, i.e. degree of $\pi_j = j$ and

$$\langle \pi_i, \pi_j \rangle = \int_a^b \pi_i(x)\pi_j(x)\omega(x)dx = \delta_{ij}.$$

Let P be a polynomial of degree n and denote by $\mathbb{P}_m$ the linear space of real polynomials of degree at most m ($m \leq n$). Consider the least squares problem

$$Q^* = \underset{Q\in\mathbb{P}_m}{\arg\min} \int_a^b (Q(x) - P(x))^2 \, \omega(x)dx, \quad m \leq n. \qquad (\bigstar)$$

Writing P as

$$P(x) = \sum_{k=0}^{n} a_k \pi_k(x),$$

show that the solution to ($\bigstar$) is given by

$$Q^*(x) = \sum_{k=0}^{m} a_k \pi_k(x).$$