Department of Mathematics and Statistics, KFUPM
Math-580-Comprehensive, Year 2020-2021

MM: 100 Duration: 120 minutes

Q1 (**5 + 5 pts**) : **(a) Let Ω_1 and Ω_2 be convex subsets of $\mathbb{R}^n$. Prove that $\Omega_1 + \Omega_2$ is a convex set of $\mathbb{R}^n$.**
(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex function and let $\psi : \mathbb{R}^n \to \bar{\mathbb{R}}$ be non-decreasing and convex on a convex set containing the range of the function f. Show that $\psi o f$ is convex.

Q2 (**12pts**) : **Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function on a nonempty convex set Ω in $\mathbb{R}^n$. Prove that the following are equivalent.**
(i) f is convex.
(ii) $f(x) - f(\bar{x}) \geq \nabla f(\bar{x})(x - \bar{x})$ **for all** $x, \bar{x} \in \Omega$

Q3 (**8pts**) : **Show that the following problem is a convex optimization problem**

$$\textbf{Minimize } f(x_1, x_2) = \sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2 + (x_3 - x_6)^2}$$

$$\textbf{subject to} \quad (x_4 - 3)^2 + x_5^2 \leq 1,$$

$$4 \leq x_6 \leq 7.$$

Q4 (**1 + 4 + 6 pts**) : **(a) Define a normal cone to a convex set at a point.**

(b) Let $\bar{x} \in \Omega$ for a convex subset Ω of $\mathbb{R}^n$. Then prove that $N(\bar{x}; \Omega)$ is a closed, convex cone containing the origin.

(c) Define domain and epigraph of an extended real-valued function f. Show that $\partial f(\bar{x}) = N((\bar{x}; dom f)\}$, where $\partial f(\bar{x})$ is the singular subdifferential of f at $\bar{x} \in dom f$.

Q5 (**5 + 5 + 4 pts**) : **(a) Determine the subdifferential of the convex function**

$$f(x) = \begin{cases} -x, & if \quad x < 0 \\ x^2 & if \quad x \geq 0 \end{cases}$$

at the points -1 and 0.

(b) Use the the definition of directional derivative to determine $\partial f(0)$, where

$$f(x_1, x_2) = |x_1| + x_2^2.$$

(c) Let $f_1(x) = -x$ **and** $f_2(x) = x, x \in R$. **use the result**

$$\partial(max f_i)(\bar{x}) \supset co\{\cap_{i \in I(\bar{x})} \partial f_i(\bar{x})\}, , i = 1, 2, \cdots, m$$

to compute $\partial f(0)$.

Or

(c) Define Gateaux derivative of an extended real-valued function at the fixed point. If f **is a Gateaux differentiable at** $\bar{x}$**, then show that** $\partial f(\bar{x})$ **is a singleton.**

Q6 (**10pts**) : **Compute the Fenchel conjugate of**

$$f(x) = \begin{cases} \frac{x^p}{p}, & if \quad x \geq 0 \\ \\ \infty & if \quad x < 0, \end{cases}$$

where $x \in R, p \in R, p > 1$.

Q7 ($\mathbf{3+7}$ **pts**) : **(a) Define polar cone, dual cone and tangent cone to a convex set at a point.**

(b) Let Ω **be a non-empty convex set in** R^n**. Let** $\bar{x} \in \Omega$**. Then prove that the normal cone of** Ω **at** $\bar{x}$ **is the polar cone of the tangent cone of** Ω **at** $\bar{x}$ **(That is** $N(\bar{x}; \Omega) = (T(\bar{x}; \Omega))^*$**).**

Or

(b) Let Ω **be a non-empty convex set in** R^n**. Let** Ω_1 **and** Ω_2 **be convex sets with** $int(\Omega_1 \bigcap \Omega_2) \neq \phi$. **Show that**

$$T(\bar{x}; \Omega_1 \bigcap \Omega_2) = T(\bar{x}; \Omega_1) \bigcap T(\bar{x}; \Omega_2)$$

for any $\bar{x} \in \Omega_1 \bigcap \Omega_2$.

Q8 (**15 pts**) : **Use Karush-Kuhn-Tucker necessary conditions to determine all solutions of the following problem:**

Maximize $f(x_1, x_2) = (x_1 + 2)^2 + (x_2 - 1)^2$

subject to $\quad -x_1 + x_2 - 2 \leq 0$

$$x_1^2 - x_2 \leq 0.$$

Q9 (**10pts**) : **Sketch the feasible set defined by**

$$S = \{(x_1, x_2) : x_2 - 2 \leq 0, \ 1 + (x_1 - 1)^2 - x_2 \leq 0, \}.$$

Find the set of the feasible direction at point $(1, 1)$ **of the feasible set** S**.**