Math640 Comprehensive Exam
KFUPM
Duration: 180 minutes
Top 9 questions will be selected for your score

Name: ________________ **ID:** _____

Question 1: Let $X, Y \in R^n$. Show that

$$|X||Y| - X \cdot Y \leq \frac{1}{2}|X - Y|^2.$$

<u>Question 2</u>: Consider

$$g(X) = \frac{x_1 x_2}{1 + x_1^2 + x_2^2} \quad for \quad X = (x_1, x_2) \in R^2.$$

Use directional derivatives to determine whether $X = O$ is a minimal, maximal, or saddle stationary point.

<u>Question 3</u>: Assume that Gateaux variations $\delta J(y;v)$ and $\delta K(y;v)$ exist for all $y, v \in Y$. Show that

$$\delta\left(\frac{J}{K}\right)(y;v) = \frac{K(y)\delta J(y;v) - J(y)\delta K(y;v)}{K(y)^2}, \quad provided\ that\ K(y) \neq 0.$$

Question 4: Let $J(y) = \int_a^b f(y(x), y'(x))\, dx$ be strictly convex on

$$D = \{y \in C^1[a,b], y(a) = a_1, y(b) = b_1\}.$$

Show that the minimizer y_0 of J satisfies the differential equation:

$$f(y, y') - y' f_{y'}(y, y') = const.$$

<u>Question 5</u>: Let $y = y(x) \geq 0$ be a smooth curve of length (ℓ) on which an object of mass (m) is sliding down from the point $y_A = y(0)$ to $y_B = y(1)$, due to the gravitational acceleration (g). The object instantly takes an amount of time dt to cut a distance ds with speed v.

Formulate the total traveling time T of the object in terms of y.

<u>Question 6</u>: The hanging cable problem suggests a minimal of the total potential energy, equivalently written as:

$$E(y) = \int_0^L y\, ds,$$

where L is the length of the cable, and $y \in C^1[-a,a] \cap \{y(-a) = y(a)\}$.

a) Develop a differential equation for finding the shape function of the cable.
b) The curve $y = c\cosh(\frac{x}{c})$ is a typical shape of the cable. Find an equation for determining the value of c.

Question 7: The total potential energy of a fluid of volume V_0, in a cylindrical, rotating column is given by

$$J(y) = \rho\pi \int_0^\ell [gy^2(x) - w^2x^2y(x)]xdx,$$

where ℓ is the radius of the column, and $y(x)$ is the fluid's level at x. The fluid's surface takes the shape $y = y(x)$ that minimizes J. Find the shape of the fluid's surface.

Question 8: Let

$$F(y) = \int_1^2 [2y(x)^2 + x^2 y'(x)^2]\, dx \quad on\ D = \{y \in C^1[1,2]: y(1) = 1\}.$$

a) Show that F has a unique minimizer.
b) Find the stationary function of F.
c) Find the minimizer of F.

Question 9: Let $f \in C^1([a,b] \times R^4)$ and

$$F(y) = \int_a^b f\big(x, y(x), y'(x), y''(x), y'''(x)\big)\, dx$$

on

$$D = \{y \in C^3[a,b]: y(a) = a_0, y'(a) = a_1, y''(a) = a_2, y(b) = b_0, y'(b) = b_1, y''(b) = b_2\}.$$

Derive the Euler-Lagrange equation for minimizing F.

Question 10: Let $f = f(z)$ be a strictly convex function on I, and

$$m = \frac{b_1 - a_1}{b - a} \in I.$$

Find the minimizer of

$$F(y) = \int_a^b f\big(y'(x)\big)dx$$

On $D = C^1[a,b] \cap \{y(a) = a_1, y(b) = b_1\}$.

<u>Question 11</u>: Consider the Dirichlet's Integral:

$$K(w) = \int_{\Omega} \left(w_x^2 + w_y^2\right) d\Omega,$$

for $w \in D = C^1(\overline{\Omega}) \cap \{w|_{\partial\Omega} = 1\}$.

a) Define an admissible domain of variations.
b) Show that K is strictly convex on D.
c) Determine an associated differential equation for finding the minimizer of K.