Department of Mathematics, KFUPM

**Control and Stability of Linear Systems**

Comprehensive Exam (221)

Duration: 150 minutes

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**Prob. 1:** (10 points) Find the characteristic polynomials and minimal polynomials of the following matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

**Prob. 2:** (20 points) The dynamics of a specific system is described by

$$\begin{cases} x_1' = x_2, \\ x_2' = x_1 - \frac{x_2^4}{x_1^2} + \sqrt{u+1}, \\ y = x_1^2 + u^2. \end{cases}$$

(a) Find all stationary points.

(b) Linearize the system around the stationary point corresponding to $u_0 = 3$.

**Prob. 3:** (20 points) Compute the transfer function of the system

$$\begin{cases} x' = Ax + bu \\ y = cx + u, \end{cases}$$

where

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad c = (1\ 1\ 1).$$

Is this system BIBO stable?

**Prob. 4:** (20 points) (a) If the exponential matrix of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix}$$

is given by

$$e^{At} = \begin{pmatrix} t+1 & t & t \\ h_1(t) & t+1 & t \\ h_2(t) & h_3(t) & -2t+1 \end{pmatrix},$$

what is $h_i(t)$, $i = 1, 2, 3$?

(b) Find a matrix $A(t)$ such that

$$\Phi(t, t_0) = \begin{pmatrix} 1 & e^{\frac{t^2 - t_0^2}{2}} - 1 \\ 0 & e^{\frac{t^2 - t_0^2}{2}} \end{pmatrix}$$

is the state transition matrix of the homogeneous ODE $x' = A(t)x$.

**Prob. 5:** (10 points) Consider the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix}$$

(a) Compute $e^{tA}$.
(b) Is the system $x' = Ax$ asymptotically stable? What about marginally stable?

**Prob. 6:** (20 points) Consider the system

$$\begin{cases} x' = (A - bk)x + bu, \\ y = cx, \end{cases}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; k = (k_1 \; k_2), \; c = (0 \; 1)$$

and $k_1$, $k_2$ are scalar constants.
(a) Compute the system's transfer function, leaving your answer as a function of the constants $k_1$ and $k_2$.
(b) Determine values for $k_1$ and $k_2$ such that the transfer function is equal to

$$\frac{s}{s^2 + s + 1}$$